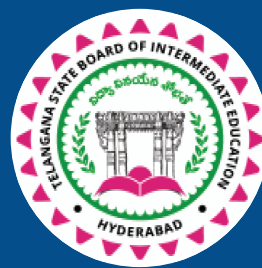
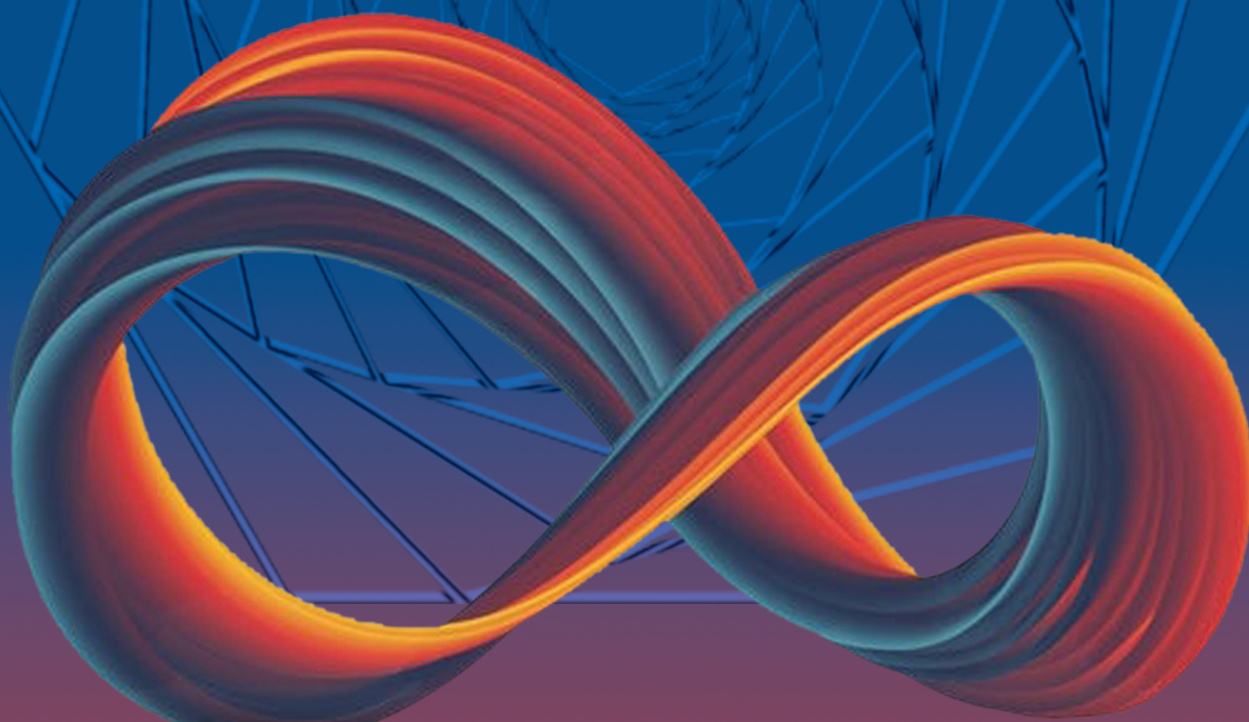


Telangana State Board of
INTERMEDIATE Education
FIRST YEAR



MATHEMATICS - I A



**BASIC
LEARNING MATERIAL**

For The Academic Year : 2021-2022



**TELANGANA STATE BOARD OF
INTERMEDIATE EDUCATION**

MATHEMATICS - IA
FIRST YEAR
(English Medium)

BASIC LEARNING MATERIAL

ACADEMIC YEAR
2021-2022

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PREFACE

The ongoing Global Pandemic Covid-19 that has engulfed the entire world has changed every sphere of our life. Education, of course is not an exception. In the absence of Physical Classroom Teaching, Department of Intermediate Education Telangana has successfully engaged the students and imparted education through TV lessons. In the back drop of the unprecedented situation due to the pandemic TSBIE has reduced the burden of curriculum load by considering only 70% syllabus for class room instruction as well as for the forthcoming Intermediate Examinations. It has also increased the choice of questions in the examination pattern for the convenience of the students.

To cope up with exam fear and stress and to prepare the students for annual exams in such a short span of time , TSBIE has prepared “Basic Learning Material” that serves as a primer for the students to face the examinations confidently. It must be noted here that, the Learning Material is not comprehensive and can never substitute the Textbook. At most it gives guidance as to how the students should include the essential steps in their answers and build upon them. I wish you to utilize the Basic Learning Material after you have thoroughly gone through the Text Book so that it may enable you to reinforce the concepts that you have learnt from the Textbook and Teachers. I appreciate ERTW Team, Subject Experts, who have involved day in and out to come out with the, Basic Learning Material in such a short span of time.

I would appreciate the feedback from all the stake holders for enriching the learning material and making it cent percent error free in all aspects.

The material can also be accessed through our website www.tsbie.cgg.gov.in.

Commissioner & Secretary
Intermediate Education, Telangana.

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FUNCTIONS

Functions: Let A and B be non-empty sets and f be a relation from A to B . If for each element $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$, then f is called a function (or) mapping from A to B . It is denoted by $f : A \rightarrow B$. The set A is called the domain of f and B is called the co-domain of f .

Range: If $f : A \rightarrow B$ is a function, then $f(A)$, the set of all f -images of elements in A , is called the range of f . Clearly $f(A) = \{f(a) / a \in A\} \subseteq B$. Also $f(A) = \{b \in B / b = f(a) \text{ for some } a \in A\}$.

Injection or one-one function: A function $f : A \rightarrow B$ is called an injection if distinct elements of A have distinct f -images in B . An injection is also called a one-one function.

$$f : A \rightarrow B \text{ is an injection} \Leftrightarrow a, a_2 \in A \text{ and } a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$$

$$\Leftrightarrow a_1, a_2 \in A \text{ and } f(a_1) = f(a_2) \Rightarrow a_1 = a_2$$

Surjection: A function $f : A \rightarrow B$ is called a surjection if the range of f is equal to the co-domain.

$$f : A \rightarrow B \text{ is a surjection} \Leftrightarrow \text{range } f = f(A) = B \text{ (co-domain)}$$

$$\Leftrightarrow B = \{f(a) \mid a \in A\}$$

$$\Leftrightarrow \text{for every } b \in B \text{ at least one } a \in A \text{ such that } f(a) = b.$$

Bijection: $f : A \rightarrow B$ is both an injection and surjection then f is said to be a bijection or one to one from A into B .

(i.e.) $f : A \rightarrow B$ is a bijection $\Leftrightarrow f$ is both injection and surjection.

$$\Leftrightarrow \text{(i) If } a_1, a_2 \in A \text{ and } f(a_1) = f(a_2) \Rightarrow a_1 = a_2$$

$$\text{(ii) for every } b \in B \exists \text{ at least one } a \in A \text{ such that } f(a) = b.$$

Finite set: If A is empty or $\exists n \in \mathbb{N}$ such that there is a bijection from A onto $\{1, 2, 3, \dots, n\}$ then A is called a finite set. In such a case we say that the number of elements in A is n and denote it by $|A|$ or $n(A)$.

Equality of functions: Let f and g be functions. We say f and g are equal and write $f = g$ if domain of f = domain of g and $f(x) = g(x)$ for all $x \in \text{domain } f$.

Identity function: Let A be a non-empty set. Then the function $f: A \rightarrow A$ defined by $f(x) = x \forall x \in A$ is called the identity function on A and is denoted by I_A .

Constant function: A function $f: A \rightarrow B$ is said to be a constant function if the range of f contains one and only one element i.e. $f(x) = c \forall x \in A$, for some fixed $c \in B$. In this case the constant function f will be denoted by C itself.

Very Short Answer Questions

1. If $f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ is defined by $f(x) = x + \frac{1}{x}$ then prove that $(f(x))^2 = f(x)^2 + f(1)$.

Sol. Since $f(x) = \left(x + \frac{1}{x}\right)$

$$\begin{aligned} f(x^2) + f(1) &= x^2 + \frac{1}{x^2} + \left(1 + \frac{1}{1}\right) \\ &= x^2 + \frac{1}{x^2} + 2 \\ &= \left(x + \frac{1}{x}\right)^2 = [f(x)]^2 \end{aligned}$$

2. If the function f is defined by $f(x) = \begin{cases} 3x - 2, & x > 3 \\ x^2 - 2, & -2 \leq x \leq 2 \\ 2x + 1, & x < -3 \end{cases}$ then find the values, if exists

(i) $f(4)$, (ii) $f(2.5)$, (iii) $f(-2)$, (iv) $f(-4)$, (v) $f(0)$, (vi) $f(-7)$.

Sol. Note that the domain of f is $(-\infty, -3) \cup [-2, 2] \cup (3, \infty)$

- (i) Since $f(x) = 3x - 2, x > 3$ we have $f(4) = 12 - 2 = 10$
(ii) 2.5 does not belong to domain f , $f(2.5)$ is not defined.
(iii) Since $f(x) = x^2 - 2, -2 \leq x \leq 2$, we have $f(-2) = (-2)^2 - 2 = 2$
(iv) Since $f(x) = 2x + 1, x < -3$, we have $f(-4) = 2(-4) + 1 = -7$
(v) Since $f(x) = x^2 - 2, -2 \leq x \leq 2$, we have $f(0) = 0^2 - 2 = -2$
(vi) Since $f(x) = 2x + 1, x < -3$, we have $f(-7) = 2(-7) + 1 = -13$

3. $A = \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$ and $f: A \rightarrow B$ is surjection defined by $f(x) = \cos(x)$ then find B .

Sol. Let $f: A \rightarrow B$ be a surjection defined by $f(x) = \cos(x)$

$$\begin{aligned}\text{Then } B = \text{rang of } f &= f(A) = \left\{ f(0), f\left(\frac{\pi}{6}\right), f\left(\frac{\pi}{4}\right), f\left(\frac{\pi}{3}\right), f\left(\frac{\pi}{2}\right) \right\} \\ &= \left[\cos 0, \cos \frac{\pi}{6}, \cos \frac{\pi}{4}, \cos \frac{\pi}{3}, \cos \frac{\pi}{2} \right] \\ &= \left\{ 1, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}, 0 \right\}\end{aligned}$$

4. Determine whether the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$ is an injection or a surjection or a bijection.

Sol. $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$ then f is not injection as

$$f(0) = \frac{e^0 - e^0}{e^0 + e^0} = 0 \text{ and } f(-1) = \frac{e^{-1} - e^{-(-1)}}{e^{-1} + e^{-(-1)}} = 0 \text{ and also } f \text{ is not a surjection since for } y = 1$$

there is no $x \in \mathbb{R} / f(x) = 1$.

If there is such $x \in \mathbb{R}$ then $e^{|x|} - e^{-x} = e^x + e^{-x}$

Clearly $x \neq 0$ for $x > 0$ this equation gives $-e^{-x} = e^{-x}$ which is not possible.

$x < 0$ this equation gives $-e^{-x} = e^x$ which is also not possible.

5. If $f(x) = \frac{\cos^2 x + \sin^4 x}{\sin^2 x + \cos^4 x} \forall x \in \mathbb{R}$ then show that $f(2012) = 1$.

$$\begin{aligned}\text{Sol. } f(x) &= \frac{\cos^2 x + \sin^4 x}{\sin^2 x + \cos^4 x} \\ &= \frac{1 - \sin^2 x + \sin^4 x}{1 - \cos^2 x + \cos^4 x} \\ &= \frac{1 - \sin^2 x (1 - \sin^2 x)}{1 - \cos^2 x (1 - \cos^2 x)} \\ &= \frac{1 - \sin^2 x \cos^2 x}{1 - \sin^2 x \cos^2 x}\end{aligned}$$

$$f(x) = 1$$

$$f(2012) = 1$$

6. If the function f is defined by $f(x) = \begin{cases} x+2, & x > 1 \\ 2 & -1 \leq x \leq 1 \\ x-1 & -3 < x < -1 \end{cases}$ then find the values of

$$(i) f(3) \quad (ii) f(0) \quad (iii) f(-1.5) \quad (iv) f(2) + f(-2) \quad (v) f(-5)$$

Sol. (i) Since $f(x) = x + 2, x > 1$ we have $f(3) = 3 + 2 = 5$

(ii) Since $f(x) = 2, -1 \leq x \leq 1$ we have $f(0) = 2$

(iii) Since $f(x) = x - 1, -3 < x < -1$ we have $f(-1.5) = -1.5 - 1 = -2.5$

(iv) Since $f(x) = x + 2, x > 1$ we have $f(2) = 2 + 2 = 4$

$f(x) = x - 1, -3 < x < -1$ we have $f(-2) = -2 - 1 = -3$

$f(2) + f(-2) = 4 + (-3) = 4 - 3 = 1$

(v) -5 does not belong to domain f , $f(-5)$ is not defined.

7. $f : R \setminus \{0\} \rightarrow R$ is defined by $f(x) = x^3 - \frac{1}{x^3}$ then show that $f(x) + f\left(\frac{1}{x}\right) = 0$.

Sol. $f(x) = x^3 - \frac{1}{x^3}$

$$f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 - \frac{1}{\left(\frac{1}{x}\right)^3}$$

$$f\left(\frac{1}{x}\right) = \frac{1}{x^3} - x^3$$

$$= f(x) + f\left(\frac{1}{x}\right) = x^3 - \frac{1}{x^3} + \frac{1}{x^3} - x^3 = 0$$

8. If $f : R \rightarrow R$ is defined by $f(x) = \frac{1-x^2}{1+x^2}$ then show that, $f(\tan \theta) = \cos 2\theta$.

Sol. $f(x) = \frac{1-x^2}{1+x^2}$

$$f(x) = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} = \frac{1 - \frac{\sin^2 \theta}{\cos^2 \theta}}{1 + \frac{\sin^2 \theta}{\cos^2 \theta}}$$

$$f(x) = \frac{\frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta}}{\frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta}} = \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta}$$

$$f(x) = \cos^2 \theta - \sin^2 \theta = \cos 2\theta$$

9. If $f : R \setminus [\pm 1] \rightarrow R$ is defined by $f(x) = \log \left| \frac{1+x}{1-x} \right|$ then show that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$.

Sol. $f(x) = \log \left| \frac{1+x}{1-x} \right|$

$$f\left(\frac{2x}{1+x^2}\right) = \log \left| \frac{1 + \frac{2x}{1+x^2}}{1 - \frac{2x}{1+x^2}} \right|$$

$$= \log \left| \frac{\frac{1+x^2+2x}{1+x^2}}{\frac{1+x^2-2x}{1+x^2}} \right|$$

$$= \log \left| \frac{(1+x)^2}{(1-x)^2} \right|$$

$$= \log \left| \frac{1+x}{1-x} \right|^2$$

$$= 2 \log \left| \frac{1+x}{1-x} \right|$$

$$f\left(\frac{2x}{1+x^2}\right) = 2f(x)$$

10. $A = \{-2, -1, 0, 1, 2\}$ and $f : A \rightarrow B$ is a surjection defined by $f(x) = x^2 + x + 1$ then find B.

Sol. $f : A \rightarrow B$ is a surjection, $\forall b \in B \exists a \in A \Rightarrow f(a) = b$

$$A = \{-2, -1, 0, 1, 2\}$$

$$f(x) = x^2 + x + 1$$

$$f(-2) = (-2)^2 + (-2) + 1 = 4 - 2 + 1 = 3$$

$$f(-1) = (-1)^2 + (-1) + 1 = 1$$

$$f(0) = (0)^2 + (0) + 1 = 1$$

$$f(1) = 1^2 + 1 + 1 = 3$$

$$f(2) = (2)^2 + 2 + 1 = 7$$

$$B = \{1, 3, 7\}$$

11. $A = \{1, 2, 3, 4\}$ and $f : A \rightarrow \mathbb{R}$ is a function defined by $f(x) = \frac{x^2 - x + 1}{x + 1}$ then find the range of f .

Sol. $f : A \rightarrow \mathbb{R} \Rightarrow f(A) = \mathbb{R}$

$$f(x) = \frac{x^2 - x + 1}{x + 1}$$

$$f(1) = \frac{1^2 - 1 + 1}{1 + 1} = \frac{1}{2}$$

$$f(2) = \frac{2^2 - 2 + 1}{2 + 1} = \frac{3}{3} = 1$$

$$f(3) = \frac{3^2 - 3 + 1}{3 + 1} = \frac{7}{4}$$

$$f(4) = \frac{4^2 - 4 + 1}{4 + 1} = \frac{13}{5}$$

$$\text{Range} = \left\{ \frac{1}{2}, 1, \frac{7}{4}, \frac{13}{5} \right\}$$

12. If the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{3^x + 3^{-x}}{2}$ then show that $f(x + y) + f(x - y) = 2f(x)f(y)$.

Sol. $f(x) = \frac{3^x + 3^{-x}}{2}$, $f(y) = \frac{3^y + 3^{-y}}{2}$

$$\text{LHS} \Rightarrow f(x + y) + f(x - y) = \frac{3^{(x+y)} + 3^{-(x+y)}}{2} + \frac{3^{x-y} + 3^{-(x-y)}}{2}$$

$$\text{LHS} \Rightarrow \frac{1}{2} [3^{x+y} + 3^{-(x+y)} + 3^{x-y} + 3^{-(x-y)}]$$

$$\text{LHS} \Rightarrow \frac{1}{2} [3^x 3^y + 3^x 3^{-y} + 3^{-x} 3^y + 3^{-x} 3^{-y}]$$

$$\text{RHS} \Rightarrow 2f(x)f(y)$$

$$= 2 \left(\frac{3^x + 3^{-x}}{2} \right) \left(\frac{3^y + 3^{-y}}{2} \right)$$

$$= \frac{1}{2} (3^x + 3^{-x}) (3^y + 3^{-y})$$

$$= \frac{1}{2} (3^x 3^y + 3^x 3^{-y} + 3^{-x} 3^y + 3^{-x} 3^{-y})$$

$$= \frac{1}{2} (3^{x+y} + 3^{x-y} + 3^{-(x-y)} + 3^{-(x+y)})$$

$$= \frac{1}{2} (3^{(x+y)} + 3^{-(x+y)} + 3^{(x-y)} + 3^{-(x-y)})$$

LHS = RHS

$$f(x+y) + f(x-y) = 2f(x)f(y)$$

PRACTICE PROBLEMS

1. $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{4^x}{4^x + 2}$ then show that $f(1-x) = 1 - f(x)$ and hence deduce the value of $f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)$.

Real valued function

If X is any set, $f : X \rightarrow \mathbb{R}$ then f is called a real valued function.

13. Find the domains of the following real valued functions

i) $f(x) = \frac{1}{(x^2 - 1)(x + 3)} \in \mathbb{R}$

$$\Rightarrow \frac{1}{(x^2 - 1)(x + 3)} \in \mathbb{R} \Rightarrow \frac{1}{(x + 1)(x - 1)(x + 3)} \in \mathbb{R}$$

$$\Rightarrow (x + 1)(x - 1)(x + 3) \neq 0 \Rightarrow x \neq -1, 1, -3,$$

Domain of f is $\mathbb{R} \setminus \{1, -1, -3\}$

ii) $f(x) = \frac{2x^2 - 5x + 7}{(x - 1)(x - 2)(x - 3)}$

$$\Rightarrow \frac{2x^2 - 5x + 7}{(x - 1)(x - 2)(x - 3)} \in \mathbb{R}$$

$$\Rightarrow (x - 1)(x - 2)(x - 3) \neq 0$$

$$\Rightarrow x \neq 1, 2, 3$$

Domain of f is $\mathbb{R} \setminus \{1, 2, 3\}$

iii) $f(x) = \frac{1}{\log(2 - x)}$

$$\frac{1}{\log(2 - x)} \in \mathbb{R}$$

$$\begin{array}{l|l} \Rightarrow (2-x) > 0 & 2-x \neq 1 \\ x-2 < 0 & -x \neq -1 \\ x < 2 & x \neq 1 \end{array}$$

Domain of f is $(-\infty, 2) - \{1\}$

iv) $f(x) = |x-3|$

$$f(x) = |x-3|$$

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\Rightarrow |x-3| = \begin{cases} x-3 & \text{if } x \geq 0 \\ -(x-3) & \text{if } x < 0 \end{cases}$$

Domain of f is $\mathbb{R}$.

v) $f(x) = \sqrt{4x-x^2}$

$$f(x) = \sqrt{4x-x^2}$$

$$= 4x - x^2 \geq 0$$

$$= x^2 - 4x \leq 0$$

$$= x(x-4) \leq 0$$

$$= x \leq 0, x-4 \leq 0$$

Domain of f is $[0, 4]$

vi) $f(x) = \frac{1}{\sqrt{1-x^2}}$

$$f(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2} > 0$$

$$\Rightarrow x^2 - 1 < 0$$

$$\Rightarrow (x+1)(x-1) < 0$$

$$\Rightarrow (x+1) < 0, (x-1) < 0$$

$$x > -1; x < 1$$

Domain of f is $(-1, 1)$

14. Find the range of the following real valued functions

i) $\log|4-x^2|$

Sol. $f(x) = \log|4 - x^2|$

$$f(x) = \log x; \text{ Range} = (-\alpha, \alpha)$$

$$f(x) = |x|; \text{ Range} = [0, \alpha)$$

$$f(x) \in R \Rightarrow 4 - x^2 \neq 0, x^2 \neq 4, x \neq -2, 2$$

$$\text{Domain of } f = R - \{-2, 2\}$$

$$\text{Range of } f = R$$

ii) $f(x) = \sqrt{[x] - x}$

Sol. $f(x) = \sqrt{[x] - x}$

$$f(x) = [x] - x \geq 0$$

$$= [x] \geq x$$

$$\text{Domain of } f = \text{Integers } Z$$

$$\text{Range of } f = \{0\}$$

iii) $f(x) = \frac{\sin \pi [x]}{1 + [x^2]}$

Sol. $f(x) = \frac{\sin \pi [x]}{1 + [x^2]}$

$$= 1 + [x^2] \neq 0$$

$$\text{Domain of } f = R$$

$$[\because \sin \pi = 0]$$

$$\text{Range of } f = \{0\}$$

iv) $f(x) = \frac{x^2 - 4}{x - 2}$

Sol. $f(x) = \frac{x^2 - 4}{x - 2}$

$$x - 2 \neq 0$$

$$f(x) = \frac{x^2 - 4}{x - 2}$$

$$= \frac{(x + 2)(x - 2)}{(x - 2)} = x + 2$$

$$f(x) \neq 2 + 2 = 4$$

$$\text{Domain of } f = R - \{2\}$$

$$\text{Range of } f = R - \{4\}$$

PRACTICE PROBLEMS**I.** Find the domains of the following real valued functions

- (i) $f(x) = \frac{3^x}{x+1}$ Ans: $\mathbb{R} - \{-1\}$
- (ii) $f(x) = \sqrt{x^2 - 25}$ Ans: $\mathbb{R} - (-5, 5)$
- (iii) $f(x) = \sqrt{x - [x]}$ Ans: $\mathbb{R}$
- (iv) $f(x) = \sqrt{[x] - x}$ Ans: $\mathbb{Z}$
- (v) $f(x) = \frac{1}{6x - x^2 + 5}$ Ans: $\mathbb{R} - \{1, 5\}$
- (vi) $f(x) = \frac{1}{\sqrt{x^2 - a^2}} (a > 0)$ Ans: $\mathbb{R} - [-a, a]$
- (vii) $f(x) = \sqrt{(x+2)(x-3)}$ Ans: $\mathbb{R} - (-2, 3)$
- (viii) $f(x) = \sqrt{(x-\alpha)(\beta-x)} \quad (0 < \alpha < \beta)$ Ans: $x \in [\alpha, \beta]$
- (ix) $f(x) = \sqrt{2-x} + \sqrt{1+x}$ Ans: $[-1, 2]$
- (x) $f(x) = \sqrt{x^2 - 1} + \frac{1}{\sqrt{x^2 - 3x + 2}}$ Ans: $\mathbb{R} - [-1, 2]$

II. Find the ranges of the following real valued functions

- (i) $\sqrt{9+x^2}$ Ans: $[3, \infty)$

దీర్ఘ సమాధాన ప్రశ్నలు**1.** If $f = \{(4,5) (5,6) (6,-4)\}$, $g = \{(4,-4) (6,5) (8,5)\}$ then find

- (i) $f + g$ (ii) $f - g$ (iii) $2f + 4g$ (iv) $f + 4$ (v) fg
 (vi) f / g (vii) $|f|$ (viii) $\sqrt{f}$ (ix) f^2 (x) f^3

Sol. Domain of f $A = \{4, 5, 6\}$ Domain of g $B = \{4, 6, 8\}$ Domain of $f + g$ $A \cap B = \{4, 6\}$

- (i) $f + g = \{(4, 5-4), (6, -4+5)\} = \{(4, 1)(6, 1)\}$
- (ii) $f - g = \{(4, 5+4), (6, -4-5)\} = \{(4, 9)(6, -9)\}$
- (iii) Domain of $2f$ $A = \{4, 5, 6\}$
 Domain of $4g$ $B = \{4, 6, 8\}$
 $\therefore 2f = \{(4, 10), (5, 12), (6, -8)\}$

$$\therefore 4g = \{(4, -16), (6, 20), (8, 20)\}$$

$$\text{Domain of } 2f + 4g = [4, 6]$$

$$2f + 4g = \{(4, 10-16), (6, -8+20)\} = \{(4, -6), (6, 12)\}$$

$$(iv) \text{ Domain of } f+4 \quad A = \{4, 5, 6\}$$

$$f+4 = \{(4, 5+4), (5, 6+4), (6, -4+4)\} \\ = \{(4, 9), (5, 10), (6, 0)\}$$

$$(v) \text{ Domain of } fg \quad A \cap B = \{4, 6\}$$

$$fg = \{(4, (5)(-4)), (6, (-4)(-5))\} \\ = \{(4, -20), (6, 20)\}$$

$$(vi) \text{ Domain of } \frac{f}{g} = \{4, 6\}$$

$$\therefore = \left\{ \left(4, \frac{-5}{4} \right) \left(6, \frac{-4}{5} \right) \right\}$$

$$(vii) \text{ Domain of } |f| \quad A = \{4, 5, 6\}$$

$$|f| = \{(4, 5), (5, 6), (6, 4)\}$$

$$(viii) \text{ Domain of } \sqrt{f} \quad \{4, 5\}$$

$$\sqrt{f} = \{(4, \sqrt{5}), (5, \sqrt{6})\}$$

$$(ix) \text{ Domain of } f^2 = A = \{4, 5, 6\}$$

$$f^2 = \{(4, 25), (5, 36), (6, 16)\}$$

$$(x) \text{ Domain of } f^3 = A = \{4, 5, 6\}$$

$$f^3 = \{(4, 125), (5, 216), (6, -64)\}$$

$$2. \quad f(x) = x^2 \text{ and } g(x) = |x| \text{ find the following functions}$$

$$(i) f+g \quad (ii) f-g \quad (iii) fg \quad (iv) 2f \quad (v) f^2 \quad (vi) f+3$$

$$\text{Sol. } f(x) = x^2$$

$$g(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$\text{Domain } f = \text{Domain of } g = \mathbb{R}$$

Hence the domain of all the functions is $\mathbb{R}$.

$$(i) \quad (f+g)(x) = f(x) + g(x) = x^2 + |x| = \begin{cases} x^2 + x, & x \geq 0 \\ x^2 - x & x < 0 \end{cases}$$

$$(ii) \quad (f-g)(x) = f(x) - g(x) = x^2 - |x| = \begin{cases} x^2 - x, & x > 0 \\ x^2 + x & x < 0 \end{cases}$$

$$(iii) \quad (fg)(x) = f(x)g(x) = x^2|x| = \begin{cases} x^3 & x \geq 0 \\ x^3 & x < 0 \end{cases}$$

$$(iv) \quad (2f)x = 2f(x) = 2x^2$$

$$(v) \quad f^2(x) = (f(x))^2 = (x^2)^2 = x^4$$

$$(vi) \quad (f+3)(x) = f(x) + 3 = x^2 + 3$$

3. If f and g are real valued functions defined by $f(x) = 2x - 1$, $g(x) = x^2$, then find

$$(i) (3f - 2g)x \quad (ii) (fg)(x) \quad (iii) \left(\frac{\sqrt{f}}{g}\right)(x) \quad (iv) (f + g + 2)(x)$$

$$\text{Sol. } f(x) = 2x - 1 \quad g(x) = x^2$$

$$\Rightarrow (f - g)x = f(x) - g(x)$$

$$(i) \quad 3f(x) - 2g(x) = 3(2x - 1) - 2(x^2) \\ = 6x - 3 - 2x^2 \\ = -2x^2 + 6x - 3$$

$$(3f - 2g)x = -2x^2 + 6x - 3$$

$$(ii) \quad (fg)(x) = f(x).g(x) \\ = (2x - 1)(x^2) = 2x^3 - x^2$$

$$(iii) \quad \left(\frac{\sqrt{f}}{g}\right)x = \frac{\sqrt{f(x)}}{g(x)} = \frac{\sqrt{2x - 1}}{x^2}$$

$$(iv) \quad (f + g + 2)x = f(x) + g(x) + 2 \\ = 2x - 1 + x^2 + 2 \\ = x^2 + 2x + 1 = (x + 1)^2$$

4. $f = \{(1, 2)(2, -3)(3, -1)\}$ then find

$$(i) 2f \quad (ii) 2 + f \quad (iii) \sqrt{f} \quad (iv) f^2$$

$$\text{Sol. } f = \{(1, 2)(2, -3)(3, -1)\}$$

$$\text{Domain of } f, A = \{1, 2, 3\}$$

$$(i) \quad 2f = \{(1, 4)(2, -6)(3, -2)\}$$

$$(ii) \quad 2 + f = \{(1, 2+2)(2, -3+2)(3, -1+2)\} \\ 2 + f = \{(1, 4)(2, -1)(3, 1)\}$$

$$(iii) \quad \sqrt{f} = \{(1, \sqrt{2})\}$$

$$(iv) \quad f^2 = \{(1, 4)(2, 9)(3, 1)\}$$



MATRICES

Matrix

An ordered rectangular array of elements is called as matrix.

$$\text{Ex: } A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$$

Order of Matrix

A matrix having m rows and n columns is said to be of order $m \times n$, read as m cross n or m by n .

1. Types of Matrices

A matrix in which the number of rows is equal to the number of columns is called a square matrix.

$$\text{Ex: } \begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix}_{2 \times 2} \quad \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 2 \\ 7 & 6 & 9 \end{bmatrix}_{3 \times 3}$$

2. Principal Diagonal / Diagonal

If $A = [a_{ij}]$ is a square matrix of order n , the elements $a_{11}, a_{12}, \dots, a_{nn}$ said to constitute its principal diagonal or simply the diagonal. Hence a_{ij} is an element of the diagonal according as $i = j$.

$$\begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 2 \\ 7 & 6 & 9 \end{bmatrix}$$

3. Trace of Matrix

The sum of the elements of the diagonal of a square matrix A is called the trace of A and is denoted by $\text{Tr}(A)$.

$$\text{Tr}(A) = \sum_{i=1}^n a_{ii}$$

Ex: If $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 2 \\ 7 & 6 & 9 \end{bmatrix}$ then $T_r(A) = 2 + (-1) + 9 = 10$

4. Diagonal Matrix

If each non-diagonal element of a square matrix is equal to zero, then the matrix is called a diagonal matrix.

Trace of Matrix

Ex: $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are diagonal matrices.

5. Scalar Matrix

If each non-diagonal element of a square matrix is zero and all diagonal elements are equal to each other, then it is called a scalar matrix.

Ex: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ are all scalar matrices.

6. Unit matrix / Identity matrix

If each non-diagonal element of a square matrix is equal to zero and each diagonal element is equal to 1 then that matrix is called a unit matrix or identity matrix.

Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$ are identity matrices.

7. Null Matrix or Zero matrix

If each element of a matrix is zero, then it is called a null matrix or zero matrix. It is denoted by $0_{m \times n}$ or 0.

Ex.: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2}$, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3 \times 2}$

8. Row matrix

A matrix with only one row is called a Row matrix.

Ex: $[1 \ 3 \ -2]_{1 \times 3}$

9. Column Matrix

A matrix with only one column is called a column matrix.

$$\text{Ex: } \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}_{3 \times 1}$$

10. Triangular matrices

A square matrix $A = [a_{ij}]$ is said to be upper triangular if $a_{ij} = 0$ for all $i > j$.

* A is said to be lower triangular if $a_{ij} = 0 \quad \forall \quad i < j$

$$\text{Ex: } \begin{bmatrix} 2 & -4 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 0 & 4 \end{bmatrix} \text{ are upper triangular matrices.}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \text{ are lower triangular matrices.}$$

Equality of matrices

Matrices A and B are said to be equal.

IF A and B are of the same order and the corresponding elements of A and B are the same.

$$\text{Thus } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

are equal if $a_{ij} = b_{ij}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$

Sum of two matrices

Let A and B be matrices of the same order. Then the sum of A and B, denoted by $A + B$ is defined as the matrix of the same order in which each element is the sum of the corresponding elements of A and B.

Scalar multiple of a matrix

Let A be a matrix of order $m \times n$ and k be a scalar. Then the $m \times n$ matrix obtained by multiplying each element of A by k is called a scalar multiple of A and is denoted by kA .

If $A = [a_{ij}]_{m \times n}$ then $kA = [ka_{ij}]_{m \times n}$

Properties of Scalar multiplication of a matrix

Let A and B be matrices of the same order and α, β be scalars. Then

$$(i) \quad \alpha (\beta A) = (\alpha\beta)A = \beta (\alpha A)$$

$$(ii) (\alpha + \beta)A = \alpha A + \beta A$$

$$(iii) OA = O$$

$$(iv) \alpha O = O$$

$$(v) \alpha(A + B) = \alpha A + \alpha B$$

Very Short Answer Questions

1. If $A = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ and $A + B = X$ then find the values of x_1, x_2, x_3, x_4 .

$$\text{Sol. } A = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

$$\Rightarrow A + B = X$$

$$= \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 \\ 7 & -3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 1 \\ x_2 = 4 \\ x_3 = 7 \\ x_4 = -3 \end{matrix}$$

2. $A = \begin{bmatrix} -1 & -2 & 3 \\ 1 & 2 & 4 \\ 2 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 5 \\ 1 & -2 & 2 \\ 1 & 2 & -3 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ then find $A + B + C$.

$$\text{Sol. } A = \begin{bmatrix} -1 & -2 & 3 \\ 1 & 2 & 4 \\ 2 & -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 & 5 \\ 1 & -2 & 2 \\ 1 & 2 & -3 \end{bmatrix}, C = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

$$A + B + C = \begin{bmatrix} -1+1+(-2) & -2+(-2)+1 & 3+5+2 \\ 1+0+1 & 2+(-2)+1 & 4+2+2 \\ 2+1+2 & -1+2+0 & 3+(-3)+1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -3 & 10 \\ 2 & 1 & 8 \\ 5 & 1 & 1 \end{bmatrix}$$

3. If $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -3 & -1 & 0 \\ 2 & 1 & 3 \\ 4 & -1 & 2 \end{bmatrix}$ and $X = A + B$ then find X.

Sol. $X = A + B$

$$X = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 0 \\ 1 & 3 & 1 \end{bmatrix} + \begin{bmatrix} -3 & -1 & 0 \\ 2 & 1 & 3 \\ 4 & -1 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} 3+(-3) & 2+(-1) & -1+0 \\ 3+2 & -2+1 & 0+3 \\ 1+4 & 3+(-1) & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -1 & 3 \\ 5 & 2 & 3 \end{bmatrix}$$

4. If $\begin{bmatrix} x-3 & 2y-8 \\ z+2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -2 & a-4 \end{bmatrix}$ then find the values of x, y, z, a .

Sol. $\begin{bmatrix} x-3 & 2y-8 \\ z+2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -2 & a-4 \end{bmatrix}$

$$\Rightarrow x-3=5 \Rightarrow x=5+3=8 \Rightarrow \boxed{x=8}$$

$$\Rightarrow 2y-8=2 \Rightarrow 2y=2+8=8$$

$$2y=10$$

$$y = \frac{10}{2} = 5 \Rightarrow \boxed{y=5}$$

$$\Rightarrow z+2=-2 \Rightarrow z=-2-2=-4 \Rightarrow \boxed{z=-4}$$

$$\Rightarrow 6=a-4 \Rightarrow a=6+4=10 \Rightarrow \boxed{a=10}$$

5. If $\begin{bmatrix} x-1 & 2 & 5-y \\ 0 & z-1 & 7 \\ 1 & 0 & a-5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 1 & 0 & 0 \end{bmatrix}$ then find the values of x, y, z, a .

Sol. $\begin{bmatrix} x-1 & 2 & 5-y \\ 0 & z-1 & 7 \\ 1 & 0 & a-5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 1 & 0 & 0 \end{bmatrix}$

$$\Rightarrow x-1=1 \Rightarrow x=1+1=2 \Rightarrow \boxed{x=2}$$

$$\Rightarrow 5 - y = 3 \Rightarrow y = 5 - 3 = 2 \Rightarrow \boxed{y = 2}$$

$$\Rightarrow z - 1 = 4 \Rightarrow z = 4 + 1 = 5 \Rightarrow \boxed{z = 5}$$

$$\Rightarrow a - 5 = 8 \Rightarrow \boxed{a = 5}$$

6. Find the trace of $A = \begin{bmatrix} 1 & 3 & -5 \\ 2 & -1 & 5 \\ 2 & 0 & 1 \end{bmatrix}$.

Sol. $A = \begin{bmatrix} 1 & 3 & -5 \\ 2 & -1 & 5 \\ 2 & 0 & 1 \end{bmatrix}$

$$\text{Trace of } A = 1 + (-1) + 1 = 1$$

7. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ then find $B - A$ and $4A - 5B$.

Sol. $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$\Rightarrow B - A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix}$$

$$B - A = \begin{bmatrix} -1 & 1 & 1 \\ -2 & -2 & -4 \\ -4 & -5 & 5 \end{bmatrix}$$

$$4A - 5B = 4 \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix} - 5 \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & 8 \\ 8 & 12 & 16 \\ 16 & 20 & -24 \end{bmatrix} - \begin{bmatrix} -5 & 10 & 15 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$4A - 5B = \begin{bmatrix} 5 & -6 & -7 \\ 8 & 7 & 16 \\ 16 & 20 & -19 \end{bmatrix}$$

8. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ then find $3B - 2A$.

Sol. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \Rightarrow 2A = \begin{bmatrix} 2 & 4 & 6 \\ 6 & 4 & 2 \end{bmatrix}$

$$B = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \Rightarrow 3B = \begin{bmatrix} 9 & 6 & 3 \\ 3 & 6 & 9 \end{bmatrix}$$

$$3B - 2A = \begin{bmatrix} 9 & 6 & 3 \\ 3 & 6 & 9 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \\ 6 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -3 \\ -3 & 2 & 7 \end{bmatrix}$$

9. If $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ then show that $A^2 = -I$ ($i^2 = -1$).

Sol. $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$$A \times A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \times \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$$A^2 = \begin{bmatrix} i^2 + 0 & 0 + 0 \\ 0 + 0 & 0 + i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^2 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \boxed{A^2 = -I}$$

10. If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ then find A^2 .

Sol. $A^2 = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 16 + (-2) & 8 + 2 \\ -4 + (-1) & -2 + 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 14 & 10 \\ -5 & -1 \end{bmatrix}$$

11. If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ then find A^2 .

Sol. $A^2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \times \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

$$A^2 = \begin{bmatrix} i^2 + 0 & 0 + 0 \\ 0 + 0 & 0 + i^2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \boxed{A^2 = -I}$$

12. If $A = \begin{bmatrix} 2 & 4 \\ -1 & k \end{bmatrix}$ and $A^2 = 0$ then find k .

Sol. $A = \begin{bmatrix} 2 & 4 \\ -1 & k \end{bmatrix}$

$$A^2 = A \times A = \begin{bmatrix} 2 & 4 \\ -1 & k \end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ -1 & k \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 + (-4) & 8 + 4k \\ -2 + (-k) & -4 + k^2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 8 + 4k \\ -2 - k & -4 + k^2 \end{bmatrix}$$

$$A^2 = 0$$

$$\begin{bmatrix} 0 & 8 + 4k \\ -2 - k & -4 + k^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$8 + 4k = 0$$

$$4k = -8$$

$$k = \frac{-8}{4} = -2 \quad \therefore k = -2$$

13. If $A = \begin{bmatrix} -2 & 1 \\ 5 & 0 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & 1 \\ 4 & 0 & 2 \end{bmatrix}$ then find $2A+B^1$ and $3B^1-A$.

Sol. $A = \begin{bmatrix} -2 & 1 \\ 5 & 0 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 3 & 1 \\ 4 & 0 & 2 \end{bmatrix}$

$$2A = 2 \begin{bmatrix} -2 & 1 \\ 5 & 0 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 10 & 0 \\ -2 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 3 & 1 \\ 4 & 0 & 2 \end{bmatrix} \quad B^1 = \begin{bmatrix} -2 & 4 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow 2A + B^1 = \begin{bmatrix} -4 & 2 \\ 10 & 0 \\ -2 & 8 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 3 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 13 & 0 \\ -1 & 10 \end{bmatrix}$$

$$\Rightarrow 3B^1 = 3 \begin{bmatrix} -2 & 4 \\ 3 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 12 \\ 9 & 0 \\ 3 & 6 \end{bmatrix}$$

$$\Rightarrow 3B^1 - A = \begin{bmatrix} -6 & 12 \\ 9 & 0 \\ 3 & 6 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 5 & 0 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 11 \\ 4 & 0 \\ 4 & 2 \end{bmatrix}$$

14. If $A = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}$ then find $A+A^1$, AA^1 .

Sol. $A = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}$, $A^1 = \begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix}$

$$A+A^1 = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -9 \\ -9 & 6 \end{bmatrix}$$

$$AA^1 = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix}$$

$$AA^1 = \begin{bmatrix} 4+16 & -10-12 \\ -10-12 & 25+9 \end{bmatrix}$$

$$AA^1 = \begin{bmatrix} 20 & -22 \\ -22 & 34 \end{bmatrix}$$

Symmetric matrix

A square matrix A is said to be symmetric if $A^1 = A$.

$$\text{Ex: } A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -3 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

Skew Symmetric matrix

A square matrix A is said to be skew symmetric if $A^1 = -A$.

$$\text{Ex: } \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$$

15. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{bmatrix}$ is a symmetric matrix, then find x.

$$\text{Sol. } A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{bmatrix}, A^1 = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & x \\ 3 & 6 & 7 \end{bmatrix}$$

Symmetric matrix $A^1 = A$

$$\begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & x \\ 3 & 6 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{bmatrix}$$

$$\boxed{x = 6}$$

16. If $A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & x & 0 \end{bmatrix}$ is a skew symmetric matrix, then find x.

Sol. Skew symmetric matrix = $A^1 = -A$

$$\begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & x \\ 1 & -2 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & x & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & x \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & 2 \\ 1 & -x & 0 \end{bmatrix}$$

$$\Rightarrow \boxed{x=2}$$

17. If $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$ then show that $AA^1 = A^1A = I$.

Sol. $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$, $A^1 = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$

$$\begin{aligned} AA^1 &= \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \alpha + \sin^2 \alpha & -\cos \alpha \sin \alpha + \sin \alpha \cos \alpha \\ -\sin \alpha \cos \alpha + \cos \alpha \sin \alpha & +\sin^2 \alpha + \cos^2 \alpha \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned}$$

$$\begin{aligned} AA^1 &= \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \cdot \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \alpha + \sin^2 \alpha & -\cos \alpha \sin \alpha + \sin \alpha \cos \alpha \\ -\sin \alpha \cos \alpha + \cos \alpha \sin \alpha & +\sin^2 \alpha + \cos^2 \alpha \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned}$$

$$\therefore \boxed{AA^1 - A^1A = I}$$

Short Answer Questions (4 marks)

1. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}$ then find AB, BA .

Sol. $AB = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{3 \times 3} \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 0-1+4 & 0+0-2 \\ 1-2+6 & -2+0-3 \\ 2-3+8 & -4+0-4 \end{bmatrix}$

$$\Rightarrow AB = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$$

$$\Rightarrow BA = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}_{3 \times 2} \times \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{3 \times 3}$$

Since the number of column of B is not equal to number of rows of A. BA is not defined.

2. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ then examine whether A and B commute

with respect to multiplication of matrices.

Sol. Both A and B are square matrices of order 3. Hence both AB, BA are defined as matrices of order 3.

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}_{3 \times 3} \times \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}_{3 \times 3}$$

$$= \begin{bmatrix} 1+0+3 & 0-2+6 & 2-4+0 \\ 2+0-1 & 0+3-2 & 4+6+0 \\ -3+0+2 & 0+1+4 & -6+2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & -2 \\ 1 & 1 & 10 \\ -1 & 5 & -4 \end{bmatrix}$$

$$\begin{aligned}
 BA &= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} \\
 &= \begin{bmatrix} 1+0-6 & -2+0+2 & 3+0+4 \\ 0+2-6 & 0+3+2 & 0-1+4 \\ 1+4+0 & -2+6+0 & 3-2+0 \end{bmatrix} \\
 &= \begin{bmatrix} -5 & 0 & 7 \\ -4 & 5 & 3 \\ 5 & 4 & 1 \end{bmatrix}_{3 \times 3}
 \end{aligned}$$

Which shows that $AB \neq BA$.

$\therefore$ A and B do not commute with respect to multiplication of matrices.

3. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}_{3 \times 3}$ then show that $A^2 - 4A - 5I = 0$.

Sol. $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$

$$\begin{aligned}
 \Rightarrow A^2 &= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} \\
 &= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}
 \end{aligned}$$

$$\Rightarrow 4A = 4 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$\Rightarrow 5I = 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Hence $A^2 - 4A - 5I$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A^2 - 4A - 5I = 0$$

4. If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$ then do AB and BA exist? If they exist find

them. Do A and B commute with respect to multiplication?

Sol. $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}_{2 \times 3}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}_{3 \times 2}$

AB multiplication matrix is 2×2 matrix

BA multiplication matrix is 3×3 matrix

$$\therefore AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2-8+6 & 3-10+15 \\ -8+8+10 & -12+10+5 \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ 10 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2-12 & -4+6 & 6+15 \\ 4-20 & -8+10 & 12+25 \\ 2-4 & -4+2 & 6+5 \end{bmatrix} = \begin{bmatrix} 10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

$AB \neq BA$, A and B are not commutative with respect to multiplication.

5. If $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ then find A^4 .

Sol. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow A^4 = \left\{ 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}^4 = 3^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^4$$

$$A^4 = 81 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

6. If $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ then find A^3 .

Sol. $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

7. If $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ then find $A^3 - 3A^2 - A - 3I$ (where I is unit matrix of order 3).

Sol. $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$

$$A^2 = A \times A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1+0+3 & -2-2-1 & 1+2+1 \\ 0+0-3 & 0+1+1 & 0-1-1 \\ 3+0+3 & -6-1-1 & 3+1+1 \end{bmatrix} = \begin{bmatrix} 4 & -5 & 4 \\ -3 & 2 & -2 \\ 6 & -8 & 5 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 4 & -5 & 4 \\ -3 & 2 & -2 \\ 6 & -8 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 4+0+12 & -8-5-4 & 4+5+4 \\ -3+0-6 & 6+2+2 & -3-2-2 \\ 6+0+15 & -12-8-5 & 6+8+5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 16 & -17 & 13 \\ -9 & 10 & -7 \\ 21 & -25 & 19 \end{bmatrix}$$

$$3A^2 = 3 \begin{bmatrix} 4 & -5 & 4 \\ -3 & 2 & -2 \\ 6 & -8 & 5 \end{bmatrix} = \begin{bmatrix} 12 & -15 & 12 \\ -9 & 6 & -6 \\ 18 & -24 & 15 \end{bmatrix}$$

$$3I = 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\Rightarrow A^3 - 3A^2 - A - 3I$$

$$= \begin{bmatrix} 16 & -17 & 13 \\ -9 & 10 & -7 \\ 21 & -25 & 19 \end{bmatrix} - \begin{bmatrix} 12 & -15 & 12 \\ -9 & 6 & -6 \\ 18 & -24 & 15 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3 \times 3} = 0$$

$$\therefore \boxed{A^3 - 3A^2 - A - 3I = 0}$$

8. If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then show that $(aI + bE)^3 = a^3I + 3a^2bE$. (Where I is unit matrix).

Sol. LHS = $(aI + bE)^3$

$$= \left[a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right]^3$$

$$= \left[\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \right]^3 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^3$$

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^2 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \times \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 + 0 & ab + ba \\ 0 + 0 & 0 + a^2 \end{bmatrix} = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^3 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^2 \times \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix} \times \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

$$\text{L.H.S.} = \begin{bmatrix} a^3 + 0 & a^2b + 2a^2b \\ 0 + 0 & 0 + a^3 \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$$

$$\text{R.H.S.} = a^3I + 3a^2bE$$

$$= a^3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3a^2b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a^3 & 0 \\ 0 & a^3 \end{bmatrix} + \begin{bmatrix} 0 & 3a^2b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$$

$$\therefore \text{L.H.S.} = \text{RHS}$$

$$\boxed{(aI + bE)^3 = a^3I + 3a^2bE}$$

9. If $\theta - \phi = \frac{\pi}{2}$ then show that $\begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} \begin{bmatrix} \cos^2 \phi & \cos \phi \sin \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix} = 0$.

Sol. $\begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} \begin{bmatrix} \cos^2 \phi & \cos \phi \sin \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix}$

$$= \begin{bmatrix} \cos^2 \theta \cos^2 \phi + \cos \theta \sin \theta \cdot \cos \phi \sin \phi & \cos^2 \theta \cdot \cos \phi \sin \phi + \cos \theta \sin \theta \cdot \sin^2 \phi \\ \cos \theta \sin \theta + \cos^2 \phi + \sin^2 \theta \cdot \cos \phi \sin \phi & \cos \theta \sin \theta \cdot \cos \phi \sin \phi + \sin^2 \theta \cdot \sin^2 \phi \end{bmatrix}$$

$$\theta - \phi = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{2} + \phi$$

$$\Rightarrow \cos \theta = \cos \left(\frac{\pi}{2} + \phi \right) = -\sin \phi$$

$$\sin \theta = \sin \left(\frac{\pi}{2} + \phi \right) = \cos \phi$$

$$= \begin{bmatrix} \sin^2 \phi \cos^2 \phi - \sin^2 \phi \cos^2 \phi & \sin^2 \phi \cos \phi \sin \phi - \sin \phi \cos \phi \sin^2 \phi \\ (-\sin \phi \cos \phi)(\cos^2 \phi) + \cos^2 \phi \cos \phi \sin \phi & -\sin \phi \cos \phi \cos \phi \sin \phi + \cos^2 \phi \sin^2 \phi \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Singular Matrix

A square matrix is said to be singular if its determinant is zero.

Non-singular Matrix

A square matrix is said to be non-singular if its determinant is non-zero.

Adjoint of a matrix

The transpose of the matrix formed by replacing the elements of a square matrix A, with the corresponding co-factors is called the adjoint of A and is denoted by Adj A.

Invertible Matrix

Let A be a square matrix, we say that A is invertible if a matrix B exists such that $AB = BA = I$, where I is the unit matrix of the same order as A and B.

10. If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is a non-singular matrix then A is invertible and $A^{-1} = \frac{\text{Adj } A}{\det A}$.

Sol. $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$\text{Adj } A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$A \cdot \text{Adj } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \times \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$\begin{bmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{bmatrix}$$

$$= \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix} = \det A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \det A \cdot I$$

Hence $\det A \neq 0$

$$A \cdot (\text{Adj } A) = \det A \cdot I$$

$$A \left(\frac{\text{Adj } A}{\det A} \right) = I$$

$$\text{Similarly } \left(\frac{\text{Adj } A}{\det A} \right) \cdot A = I$$

$$\text{Let } B = \frac{\text{Adj } A}{\det A} \text{ then } AB = BA = I$$

$$\text{Hence } A \text{ is invertible and } A^{-1} = B = \frac{\text{Adj } A}{\det A}$$

Long Answer Questions (7 Marks)

1. Find the adjoint and the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$.

Sol. $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$\det A = 1(16 - 9) - 3(4 - 3) + 3(3 - 4) \\ = 7 - 3 - 3 = 1 \neq 0$$

$\therefore A$ is invertible.

The cofactor matrix of A is $B = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$$\text{Adj } A = B^T = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{\det A} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \because \det A = 1$$

2. Show that $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ is non-singular and find A^{-1} .

Sol. $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

$$\det A = 1(4 - 3) - 2(6 - 3) + 1(3 - 2) \\ = 1 - 6 + 1 = -4 \neq 0$$

Hence A is a non-singular matrix.

The cofactor matrix of A is $B = \begin{bmatrix} 1 & -3 & 1 \\ -3 & 1 & 1 \\ 4 & 0 & -4 \end{bmatrix}$

$$\text{Adj } A = B^T = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{\det A} = \frac{1}{-4} \begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix}$$

3. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$ then find $(A')^{-1}$.

Sol. $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$

$$A^1 = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

$$\det(A^1) = 1(-1-8) + 0 - 2(-8+3) \\ -9 + 0 + 10 = 1 \neq 0$$

$$\text{Cofactor matrix of } A^1 = \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & -6 & -1 \end{bmatrix}$$

$$\text{Adjoint matrix of } A^1 = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & -6 \\ -5 & -4 & -1 \end{bmatrix}$$

4. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ then show that $\text{adj}A = 3A^1$. Find A^{-1} .

Sol. $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$A^1 = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$3A^1 = 3 \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$3A^1 = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$\text{Cofactor matrix of } A = \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}$$

$$\text{Adj}A = \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$\therefore \text{Adj } A = 3A^T$$

$$\det A = -1(1-4) + 2(2+4) - 2(-4-2)$$

$$= 3 + 12 + 12 = 27 \neq 0$$

$$A^{-1} = \frac{\text{Adj } A}{\det A} = \frac{1}{27} \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{3}{27} & \frac{6}{27} & \frac{6}{27} \\ -\frac{6}{27} & \frac{3}{27} & -\frac{6}{27} \\ -\frac{6}{27} & \frac{6}{27} & \frac{3}{27} \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$\therefore \boxed{A^{-1} = A^1}$$

5. If $3A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ then show that $A^{-1} = A^T$.

Sol. $3A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

$$A = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

$$AA^1 = I \Rightarrow A^{-1} = A^1$$

$$A^1 = \frac{1}{3} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$A \times A^T = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \times \frac{1}{3} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 1+4+4 & 2+2-4 & -2+4-2 \\ 2+2-4 & 4+1+4 & -4+2+2 \\ -2+4-2 & -4+2+2 & 4+4+1 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$AA^1 = I$$

$$A^{-1} = A^1$$

Solution of Simultaneous Linear Equations

Cramer's Rule

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

$$\Rightarrow \text{Where } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ is non-singular matrix}$$

$$\text{Let } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ be the solution of the equation } AX = D \text{ where } D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\Rightarrow \text{Let } A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{Then } x\Delta = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 + yC_2 + zC_3$ we get

$$x\Delta = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\therefore \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \text{ then } x = \frac{\Delta_1}{\Delta}$$

$$\text{Similarly } \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ then } y = \frac{\Delta_2}{\Delta}$$

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \text{ then } z = \frac{\Delta_3}{\Delta}$$

$$\therefore \frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{1}{\Delta}. \text{ This is known as Cramer Rule.}$$

Matrix Inversion Method

Consider the matrix equation $AX = D$, where A is non-singular.

Then we can find A^{-1} .

$$AX = D \Leftrightarrow A^{-1}(AX) = A^{-1}D$$

$$(A^{-1}A)X = A^{-1}D$$

$$IX = A^{-1}D$$

$$X = A^{-1}D. \text{ From this } x, y \text{ and } z \text{ are known.}$$

6. Solve the following simultaneous linear equations by using Cramer's rule.

$$3x + 4y + 5z = 18, 2x - y + 8z = 13 \quad 5x - 2y + 7z = 20$$

Sol. $3x + 4y + 5z = 18,$

$$2x - y + 8z = 13$$

$$5x - 2y + 7z = 20$$

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad D = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

Then we can write the given equations in the form of matrix equation as $AX = D$.

$$\begin{aligned} \Delta = \det A &= \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{vmatrix} \\ &= 3(-7+16) - 4(14-40) + 5(-4+5) \\ &= 3(9) - 4(-26) + 5(1) \\ &= 27 + 104 + 5 = 136 \neq 0 \end{aligned}$$

Hence we can solve the given equation by using Crammer's rule.

$$\Delta_1 = \begin{vmatrix} 18 & 4 & 5 \\ 13 & -1 & 8 \\ 20 & -2 & 7 \end{vmatrix} = 408$$

$$\Delta_2 = \begin{vmatrix} 3 & 18 & 5 \\ 2 & 13 & 8 \\ 5 & 20 & 7 \end{vmatrix} = 136$$

$$\Delta_3 = \begin{vmatrix} 3 & 4 & 18 \\ 2 & -1 & 13 \\ 5 & -2 & 20 \end{vmatrix} = 136$$

Hence by Crammer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{408}{136} = 3$$

$$y = \frac{\Delta_2}{\Delta} = \frac{136}{136} = 1$$

$$z = \frac{\Delta_3}{\Delta} = \frac{136}{136} = 1$$

The solution of the given system of equations is $\boxed{x=3, y=1, z=1}$

7. Solve the following system of equations by Crammer's rule.

$$(i) 5x - 6y + 4z = 15, 7x + 4y - 3z = 19, 2x + y + 6z = 46$$

Sol. (i) $5x - 6y + 4z = 15,$

$$7x + 4y - 3z = 19,$$

$$2x + y + 6z = 46$$

$$\det A = \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix}, D = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix} X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\det A = \Delta = \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix} = 5(24 + 3) + 6(42 + 6) + 4(7 - 8)$$

$$= 135 + 288 - 4$$

$$\Delta = 419 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 15 & -6 & 4 \\ 19 & 4 & -3 \\ 46 & 1 & 6 \end{vmatrix} = 1257$$

$$\Delta_2 = \begin{vmatrix} 5 & 15 & 4 \\ 7 & 19 & -3 \\ 2 & 46 & 6 \end{vmatrix} = 1676$$

$$\Delta_3 = \begin{vmatrix} 5 & -6 & 15 \\ 7 & 4 & 19 \\ 2 & 1 & 46 \end{vmatrix} = 2514$$

From Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{1257}{419} = 3$$

$$y = \frac{\Delta_2}{\Delta} = \frac{1676}{419} = 4$$

$$z = \frac{\Delta_3}{\Delta} = \frac{2514}{419} = 6$$

$$\therefore \boxed{x=3, y=4, z=6}$$

8. $x + y + z = 1$

$$2x + 2y + 3z = 6$$

$$x + 4y + 9z = 3$$

Sol. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\det A = \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1(18 - 12) - 1(18 - 3) + 1(8 - 2)$$

$$\Delta = 6 - 15 + 6 = -3 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 6 & 2 & 3 \\ 3 & 4 & 9 \end{vmatrix} = -21$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 9 \end{vmatrix} = 30$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 6 \\ 1 & 4 & 3 \end{vmatrix} = -12$$

From Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-21}{-3} = 7$$

$$y = \frac{\Delta_2}{\Delta} = \frac{30}{-3} = -10$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-12}{-3} = 4$$

$$\therefore \boxed{x=7, y=-10, z=4}$$

9. $x - y + 3z = 5$

$$4x + 2y - z = 0$$

$$-x + 3y + z = 5$$

Sol. $A = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ -1 & 3 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\begin{aligned} \det A = \Delta &= \begin{vmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ -1 & 3 & 1 \end{vmatrix} = 1(2+3) + 1(4-1) + 3(12+2) \\ &= 5 + 3 + 42 = 50 \neq 0 \end{aligned}$$

$$\Delta_1 = \begin{vmatrix} 5 & -1 & 3 \\ 0 & 2 & -1 \\ 5 & 3 & 1 \end{vmatrix} = 0$$

$$\Delta_2 = \begin{vmatrix} 1 & 5 & 3 \\ 4 & 0 & -1 \\ -1 & 5 & 1 \end{vmatrix} = 50$$

$$\Delta_3 = \begin{vmatrix} 1 & -1 & 5 \\ 4 & 2 & 0 \\ -1 & 3 & 5 \end{vmatrix} = 100$$

From Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{0}{50} = 0$$

$$y = \frac{\Delta_2}{\Delta} = \frac{50}{50} = 1$$

$$z = \frac{\Delta_3}{\Delta} = \frac{100}{50} = 2$$

$$\therefore \boxed{x = 0, y = 1, z = 2}$$

10. $x + y + z = 9$

$$2x + 5y + 7z = 52$$

$$2x + y - z = 0$$

Sol. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\det A = \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix} = 1(-5 - 7) - 1(-2 - 14) + 1(2 - 10)$$

$$= -12 + 16 - 8 = -4 \neq 0$$

$$\Delta = -4$$

$$\Delta_1 = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix} = -4$$

$$\Delta_2 = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix} = -12$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix} = -20$$

From Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-4}{-4} = 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-12}{-4} = 3$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-20}{-4} = 5$$

$$\therefore x = 1, y = 3, z = 5$$

11. Solve the following systems of equations by using matrix inversion method.

i) $3x + 4y + 5z = 18, 2x - y - 8z = 13, 5x - 2y + 7z = 20$

Sol. $3x + 4y + 5z = 18$
 $2x - y - 8z = 13$
 $5x - 2y + 7z = 20$

Let $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$

Then we can write the given equations in the form

$$A X = D$$

$$\det A = \Delta = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{vmatrix} = 3(-7 + 16) - 4(14 - 40) + 5(-4 + 5)$$

$$= 27 + 104 + 5 = 136 \neq 0$$

$$\text{Cofactor matrix of } A = \begin{bmatrix} 9 & 26 & 1 \\ -38 & -4 & 26 \\ 37 & -14 & -11 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$$

$$X = A^{-1} D$$

$$X = \left(\frac{\text{Adj } A}{\det A} \right) \cdot D$$

$$= \frac{1}{136} \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix} \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

$$= \frac{1}{136} \begin{bmatrix} 408 \\ 136 \\ 136 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \boxed{x=3, y=1, z=1}$$

12. $2x - y + 3z = 9$

$$x + y + z = 6$$

$$x - y + z = 2$$

Sol. $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$

$$\boxed{AX = D \Rightarrow X = A^{-1}D}$$

$$\det A = \Delta = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2(1+1) + 1(0-0) + 3(-1-1)$$

$$= 4 - 6 = -2 \neq 0$$

$$\text{Cofactor matrix of } A = \begin{bmatrix} 2 & 0 & -2 \\ -2 & -1 & 1 \\ -4 & 1 & 3 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$$

$$X = A^{-1} D$$

$$= \left(\frac{\text{Adj } A}{\det A} \right) \cdot D$$

$$X = \frac{1}{-2} \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$$X = \frac{1}{-2} \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore x=1, y=2, z=3$$

13. $x + y + z = 1$
 $2x + 2y + 3z = 6$
 $x + 4y + 9z = 3$

Sol. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$

$$AX = D \Rightarrow X = A^{-1}D$$

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1(18 - 12) - 1(18 - 3) + 1(8 - 2)$$

$$= 6 - 15 + 6 = -3 \neq 0$$

$$\det A \neq 0 = -3$$

$$\text{Cofactor matrix of } A = \begin{bmatrix} 6 & -15 & 6 \\ -5 & 8 & -3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} 6 & -5 & 1 \\ -15 & 8 & -1 \\ 6 & -3 & 0 \end{bmatrix}$$

$$X = A^{-1} D$$

$$X = \left(\frac{\text{Adj } A}{\det A} \right) \cdot D$$

$$X = -\frac{1}{3} \begin{bmatrix} 6 & -5 & 1 \\ -15 & 8 & -1 \\ 6 & -3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -21 \\ 30 \\ -12 \end{bmatrix} = \begin{bmatrix} 7 \\ -10 \\ 4 \end{bmatrix}$$

$$\therefore x=7, y=-10, z=4$$

14. $2x - y + 3z = 8$

$$-x + 2y + z = 4$$

$$3x + y - 4z = 0$$

Sol. $A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$

$$AX = D \Rightarrow X = A^{-1}D$$

$$\det A = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{vmatrix} = 2(-8 - 1) + 1(4 - 3) + 3(-1 - 6)$$

$$= -18 + 1 - 21 = -38 \neq 0$$

$$\det A = -38 \neq 0$$

$$\text{Cofactor matrix of } A = \begin{bmatrix} -9 & -1 & -7 \\ -1 & -14 & -5 \\ -7 & -5 & 3 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} -9 & -1 & -7 \\ -1 & -14 & -5 \\ -7 & -5 & 3 \end{bmatrix}$$

$$X = A^{-1} D$$

$$X = \left(\frac{\text{Adj } A}{\det A} \right) \cdot D$$

$$X = -\frac{1}{38} \begin{bmatrix} -9 & -1 & -7 \\ -1 & -14 & -5 \\ -7 & -5 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

$$X = -\frac{1}{38} \begin{bmatrix} -76 \\ -76 \\ -76 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore x=1, y=1, z=1$$

PRACTISE PROBLEMS

1. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ then show that $A^{-1} = A^3$.
2. Solve the following system of equations by Crammer's rule.
 - (i) $2x - y + 3z = 9, x + y + z = 6, x - y + z = 2$
 - (ii) $2x - y + 3z = 8, -x + 2y + z = 4, 3x + y - 4z = 0$
 - (iii) $2x - y + 8z = 13, 3x + 4y + 5z = 18, 5x - 2y + 7z = 20$
2. Solve the following system of equations by matrix inversion method.
 - (i) $x + y + z = 1, 2x + 2y + 3z = 6, x + 4y + 9z = 3$
 - (ii) $x - y + 3z = 5, 4x + 2y - z = 0, -x + 3y + z = 5$
 - (iii) $x + y + z = 9, 2x + 5y + 7z = 52, 2x + y - z = 0$



Addition of Vectors

- ❖ **Vector:** A physical quantity which has both magnitude and direction is called a vector.
Eg.: Velocity, displacement, force etc.
- ❖ **Scalar:** A physical quantity which has only magnitude is called a scalar.
Eg.: length, volume, temperature
- ❖ **Position Vector:** : Let 'O' and 'P' be any two points in space. Then the vector $\overrightarrow{OP}$ having 'O' and 'P' as initial and terminal points respectively, is called the position vector of the point P with respect to 'O'.
Position vector of P (x,y,z) w.r.t. origin O (0, 0, 0) is denoted by $\vec{r}$.

Magnitude of $\overrightarrow{OP}$ is given by, $|\overrightarrow{OP}| = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

Note: $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ = Position vector of B – Position vector of A.

- ❖ **Direction Cosines and Direction Ratios:**

Let the position vector of point P (x,y,z) w.r.t. origin 'O' be $\overrightarrow{OP} = \vec{r}$. Let α, β, γ be the angles made by the vector $\vec{r}$ in the positive direction (counter clockwise direction) of X, Y, Z axes respectively.

Then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines of the vector $\vec{r}$.

These direction cosines are denoted by l, m, n respectively.

$$\begin{aligned} \text{i.e.} \quad l &= \cos\alpha \\ m &= \cos\beta \\ n &= \cos\gamma \end{aligned}$$

Thus the coordinates x, y, z of the point P are expressed as (lr, mr, nr) .

The numbers lr, mr, nr which are proportional to the direction cosines l, m, n are called the direction ratios of the vector $\vec{r}$. These direction ratios are denoted by a, b, c .

$$\begin{aligned} \text{i.e.} \quad a &= lr \\ b &= mr \\ c &= nr \end{aligned}$$

Note: $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$, in general.

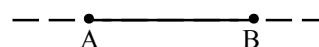
- ❖ **Unit Vector:** A vector whose magnitude is unity (i.e. 1 unit) is called a unit vector. It is represented by $\bar{e}$.
- ❖ Unit vector in the direction of a given vector $\bar{a}$ is denoted by $\hat{a}$ and it is given by,

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$
- ❖ The zero vector is denoted by $\bar{0}$ and it is also known as null vector. We can observe that the initial and terminal points coincide for zero vector and its magnitude is the scalar 0.
- ❖ **Like vectors:** If two vectors are having the same direction, then they are called like vectors.
- ❖ **Unlike vectors:** If two vectors are in opposite directions, then they are called unlike vectors.
- ❖ **Negative of a vector:** Let $\bar{a}$ be a vector. The vector having the same magnitude as $\bar{a}$ but having the opposite direction is called the negative vector of $\bar{a}$ and is denoted by $-\bar{a}$.

Note:

1. If $\bar{a} = \overline{AB}$, then $-\bar{a} = \overline{BA}$.

2. Unit vector in the opposite direction of $\bar{a} = \frac{-\bar{a}}{|\bar{a}|}$



- ❖ The line AB is called support of the vector $\overline{AB}$.
- ❖ **Collinear (Parallel) Vectors:** Vectors with same support or parallel supports are called collinear or parallel vectors.

Note: 1. $\bar{a}$, $\bar{b}$ are collinear (parallel) vectors $\Leftrightarrow \bar{a} = \lambda \bar{b}$, where λ is a scalar.

2. The points A, B, C are collinear $\Leftrightarrow \overline{AB} = \lambda \overline{BC}$, where λ is a scalar.

3. If $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are collinear vectors, then $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

- ❖ **Coplanar Vectors:** Vectors whose supports are in the same plane or parallel to the same plane are called coplanar vectors.

Note: 1. The points A, B, C, D are coplanar $\Leftrightarrow \overline{AD} = x\overline{AB} + y\overline{AC}$ where x, y are scalars.

2. If $\overline{AB} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$

$$\overline{AC} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$$

$$\overline{AD} = a_3\mathbf{i} + b_3\mathbf{j} + c_3\mathbf{k}, \text{ then the points A, B, C, D or}$$

$$\overline{AB}, \overline{AC}, \overline{AD} \text{ are coplanar} \Leftrightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

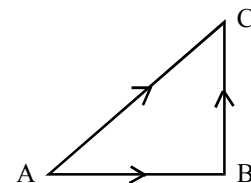
- ❖ The vectors which are not coplanar are called non-coplanar vectors.

❖ **Triangle law of vector addition:**

In $\triangle ABC$, $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are two sides,
then their sum is represented by the third side, $\overrightarrow{AC}$.

i.e. $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

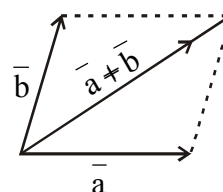
This is known as the triangle law of vector addition.



❖ **Parallelogram law of vector addition:**

If $\vec{a}$, $\vec{b}$ are the adjacent sides of a parallelogram then their sum
 $\vec{a} + \vec{b}$ is represented by the diagonal of the parallelogram through
their common point.

This is known as the parallelogram law of vector addition.



❖ **Properties of vector addition:** For any vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

(i) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (Commutative property)

(ii) $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$ (Associative property)

(iii) $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$ (Identity property)

Here, the zero vector $\vec{0}$ is called the additive identity for the vector addition.

❖ Let $\vec{a}, \vec{b}$ be two vectors, then

(i) $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$

(ii) $||\vec{a}| - |\vec{b}|| \leq |\vec{a} - \vec{b}|$

Note: Equality holds if and only if $\vec{a}$ and $\vec{b}$ are like vectors.

❖ If a point P divides the line segment joining the points A($\vec{a}$) and B($\vec{b}$) in the ratio $m : n$,
then the position vector of P is $\frac{m\vec{b} + n\vec{a}}{m + n}$.

❖ **Linear combination of vectors:** Let $\vec{a}_1, \vec{a}_2, \vec{a}_3, \dots, \vec{a}_n$ be vectors and $x_1, x_2, x_3, \dots, x_n$ be scalars. Then the vector $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + \dots + x_n\vec{a}_n$ is called a linear combination of the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \dots, \vec{a}_n$.

❖ Vector equation of the straight line passing through the point A($\vec{a}$) and parallel to the vector $\vec{b}$ is $\vec{r} = \vec{a} + t\vec{b}$, $t \in \mathbb{R}$

❖ Vector equation of the straight line passing through two points A($\vec{a}$) and B($\vec{b}$) is,
 $\vec{r} = (1-t)\vec{a} + t\vec{b}$, $t \in \mathbb{R}$

❖ Vector equation of the plane passing through a point A($\vec{a}$) and parallel to the vectors $\vec{b}, \vec{c}$ is $\vec{r} = \vec{a} + t\vec{b} + s\vec{c}$, $t, s \in \mathbb{R}$

- ❖ Vector equation of the plane passing through three points $A(\bar{a})$, $B(\bar{b})$ and parallel to the vector $\bar{c}$ is $\bar{r} = (1-t)\bar{a} + t\bar{b} + s\bar{c}$, $t, s \in \mathbb{R}$
- ❖ Vector equation of the plane passing through three points $A(\bar{a})$, $B(\bar{b})$ and $C(\bar{c})$ is $\bar{r} = (1-t-s)\bar{a} + t\bar{b} + s\bar{c}$, $t, s \in \mathbb{R}$

VERY SHORT ANSWER TYPE QUESTIONS (2 MARKS)

- 1) Find the unit vector in the direction of vector $\bar{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Sol. $\bar{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$$|\bar{a}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

$$\therefore \text{Unit vector in the direction of } \bar{a} \text{ is } \hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{2\mathbf{i} + 3\mathbf{j} + \mathbf{k}}{\sqrt{14}}$$

$$\Rightarrow \hat{a} = \frac{2}{\sqrt{14}}\mathbf{i} + \frac{3}{\sqrt{14}}\mathbf{j} + \frac{1}{\sqrt{14}}\mathbf{k}$$

- 2) Let $\bar{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\bar{b} = 3\mathbf{i} + \mathbf{j}$. Find the unit vector in the direction of $\bar{a} + \bar{b}$.

Sol. $\bar{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$$\bar{b} = 3\mathbf{i} + \mathbf{j}$$

$$\bar{a} + \bar{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + 3\mathbf{i} + \mathbf{j}$$

$$\therefore \bar{a} + \bar{b} = 4\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

$$|\bar{a} + \bar{b}| = \sqrt{4^2 + 3^2 + 3^2} = \sqrt{16 + 9 + 9} = \sqrt{34}$$

$$\therefore \text{Unit vector in the direction of } |\bar{a} + \bar{b}| = \frac{\bar{a} + \bar{b}}{|\bar{a} + \bar{b}|} = \frac{4\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}}{\sqrt{34}}$$

$$= \frac{1}{\sqrt{34}}(4\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$$

- 3) Find the unit vector in the direction of the sum of the vectors $\bar{a} = 2\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ and $\bar{b} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

Sol. $\bar{a} = 2\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$, $\bar{b} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

$$\bar{a} + \bar{b} = 2\mathbf{i} + 2\mathbf{j} - 5\mathbf{k} + 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

$$\bar{a} + \bar{b} = 4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$

$$|\bar{a} + \bar{b}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{16 + 9 + 4} = \sqrt{29}$$

$$\therefore \text{Unit vector in the direction of sum of } \vec{a} \text{ and } \vec{b} = \frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} = \frac{4\vec{i} + 3\vec{j} - 2\vec{k}}{\sqrt{29}}$$

- 4) Let $\vec{a} = 2\vec{i} + 4\vec{j} - 5\vec{k}$, $\vec{b} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{c} = \vec{j} + 2\vec{k}$. Find the unit vector in the opposite direction of $\vec{a} + \vec{b} + \vec{c}$.

Sol. $\vec{a} = 2\vec{i} + 4\vec{j} - 5\vec{k}$

$$\vec{b} = \vec{i} + \vec{j} + \vec{k}$$

$$\vec{c} = \vec{j} + 2\vec{k}$$

$$\vec{a} + \vec{b} + \vec{c} = (2\vec{i} + 4\vec{j} - 5\vec{k}) + (\vec{i} + \vec{j} + \vec{k}) + (\vec{j} + 2\vec{k})$$

$$\vec{a} + \vec{b} + \vec{c} = 3\vec{i} + 6\vec{j} - 2\vec{k}$$

$$|\vec{a} + \vec{b} + \vec{c}| = \sqrt{3^2 + 6^2 + (-2)^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = 7$$

$$\therefore \text{Unit vector in the opposite direction of } \vec{a} + \vec{b} + \vec{c}$$

$$= -\frac{(\vec{a} + \vec{b} + \vec{c})}{|\vec{a} + \vec{b} + \vec{c}|}$$

$$= -\frac{(3\vec{i} + 6\vec{j} - 2\vec{k})}{7}$$

- 5) If the position vectors of the points A, B and C are $-2\vec{i} + \vec{j} - \vec{k}$, $-4\vec{i} + 2\vec{j} + 2\vec{k}$ and $6\vec{i} - 3\vec{j} - 13\vec{k}$ respectively and $\vec{AB} = \lambda \vec{AC}$, then find the value of λ .

Sol. Let 'O' be the origin.

Then, $\vec{OA} = -2\vec{i} + \vec{j} - \vec{k}$

$$\vec{OB} = -4\vec{i} + 2\vec{j} + 2\vec{k}$$

$$\vec{OC} = 6\vec{i} - 3\vec{j} - 13\vec{k}$$

$$\therefore \vec{AB} = \vec{OB} - \vec{OA} = (-4\vec{i} + 2\vec{j} + 2\vec{k}) - (-2\vec{i} + \vec{j} - \vec{k})$$

$$= -4\vec{i} + 2\vec{j} + 2\vec{k} + 2\vec{i} - \vec{j} + \vec{k}$$

$$\therefore \vec{AB} = -2\vec{i} + \vec{j} + 3\vec{k}$$

$$\therefore \vec{AC} = \vec{OC} - \vec{OA} = (6\vec{i} - 3\vec{j} - 13\vec{k}) - (-2\vec{i} + \vec{j} - \vec{k})$$

$$= 6\vec{i} - 3\vec{j} - 13\vec{k} + 2\vec{i} - \vec{j} + \vec{k}$$

$$= 8\vec{i} - 4\vec{j} - 12\vec{k}$$

$$\vec{AC} = -4(-2\vec{i} + \vec{j} + 3\vec{k})$$

$$\vec{AC} = -4 \cdot \vec{AB} \quad \left[\because \vec{AB} = -2\vec{i} + \vec{j} + 3\vec{k} \right]$$

$$\Rightarrow -4\vec{AB} = \vec{AC}$$

$$\overline{AB} = -\frac{1}{4} \overline{AC}$$

Comparing with, $\overline{AB} = \lambda \overline{AC}$ we get,

$$\lambda = -\frac{1}{4}$$

- 6) If $\overline{OA} = i + j + k$, $\overline{AB} = 3i - 2j + k$, $\overline{BC} = i + 2j - 2k$ and $\overline{CD} = 2i + j + 3k$, then find the vector $\overline{OD}$.

Sol. $\overline{OA} = i + j + k$

$$\overline{AB} = 3i - 2j + k$$

$$\overline{BC} = i + 2j - 2k$$

$$\overline{CD} = 2i + j + 3k$$

$$\because \overline{OA} + \overline{AB} + \overline{BC} + \overline{CD} = \overline{OD}$$

$$\Rightarrow \overline{OD} = \overline{OA} + \overline{AB} + \overline{BC} + \overline{CD}$$

$$= (i + j + k) + (3i - 2j + k) + (i + 2j - 2k) + (2i + j + 3k)$$

$$\therefore \overline{OD} = 7i + 2j + 3k$$

- 7) Write direction ratios of the vector $\vec{a} = i + j - 2k$ and hence calculate its direction cosines.

Sol. Let $\vec{r} = \vec{a} = i + j - 2k$

Let $\bar{a}, \bar{b}, \bar{c}$ be the direction ratios of vector $\vec{r} = xi + yj + zk$

Then the values of a, b, c are just the respective components x, y and z of the vector.

Hence, $a = 1, b = 1, c = -2$

If l, m, n are the direction cosines of the given vector, then

$$|\vec{r}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{1 + 1 + 4} = \sqrt{6}$$

$$l = \frac{a}{|\vec{r}|} = \frac{1}{\sqrt{6}}$$

$$m = \frac{b}{|\vec{r}|} = \frac{1}{\sqrt{6}}$$

$$n = \frac{c}{|\vec{r}|} = \frac{-2}{\sqrt{6}}$$

$\therefore$ The direction cosines are $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}} \right)$

8) If the vectors $-3\mathbf{i} + 4\mathbf{j} + \lambda \mathbf{k}$ and $\mu \mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$ are collinear vectors, then find λ and μ .

Sol. The vectors, $-3\mathbf{i} + 4\mathbf{j} + \lambda \mathbf{k}$ and $\mu \mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$ are collinear.

$$\frac{-3}{\mu} = \frac{4}{8} = \frac{\lambda}{6}$$

$$\Rightarrow \frac{-3}{\mu} = \frac{1}{2} = \frac{\lambda}{6}$$

$$\Rightarrow \frac{-3}{\mu} = \frac{1}{2} \quad \text{and} \quad \frac{1}{2} = \frac{\lambda}{6}$$

$$\Rightarrow \mu = 2(-3) \quad 2\lambda = 6(1)$$

$$\Rightarrow \mu = -6 \quad \lambda = \frac{6}{2} = 3$$

$$\therefore \lambda = 3 \quad \text{and} \quad \mu = -6$$

9) Find the vector equation of the line passing through the point $2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and parallel to the vector $4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

Sol. Let $\vec{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$$\vec{b} = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$$

Vector equation of the line passing through $\vec{a}$ and parallel to $\vec{b}$ is,

$$\vec{r} = \vec{a} + t\vec{b}, \quad t \in \mathbb{R}$$

$$\vec{r} = (2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) + t(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$$

$$\Rightarrow \vec{r} = (2 + 4t)\mathbf{i} + (3 - 2t)\mathbf{j} + (1 + 3t)\mathbf{k}$$

10) OABC is a parallelogram. If $\vec{OA} = \vec{a}$ and $\vec{OC} = \vec{c}$, find the vector equation of the side $\vec{BC}$.

Sol. OABC is a parallelogram in which,

$$\vec{OA} = \vec{a}$$

$$\vec{OC} = \vec{c} \Rightarrow \vec{AB} = \vec{c}$$

$$\Rightarrow \vec{OB} - \vec{OA} = \vec{c}$$

$$\Rightarrow \vec{OB} = \vec{c} + \vec{OA}$$

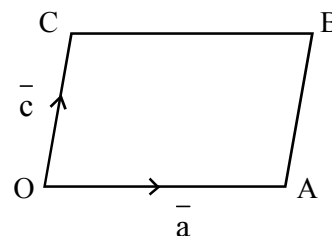
$$\Rightarrow \vec{OB} = \vec{c} + \vec{a}$$

$$\therefore \vec{OB} = \vec{a} + \vec{c}$$

$$\therefore \text{The vector equation of } \vec{BC}, \vec{r} = (1-t)\vec{c} + t(\vec{a} + \vec{c}), \quad t \in \mathbb{R}$$

$$\Rightarrow \vec{r} = (1-t+t)\vec{c} + t\vec{a}$$

$$\Rightarrow \vec{r} = \vec{c} + t\vec{a}$$



11) Find the vector equation of the line joining the points $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $-4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

Sol. Let $\vec{a} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

$$\vec{b} = -4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

Vector equation of line passing through $\vec{a}$ and $\vec{b}$ is

$$\vec{r} = (1-t)\vec{a} + t\vec{b}, t \in \mathbb{R}$$

$$\Rightarrow \vec{r} = (1-t)(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(-4\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$

$$\Rightarrow \vec{r} = (2-2t-4t)\mathbf{i} + (1-t+3t)\mathbf{j} + (3-3t-t)\mathbf{k}$$

$$\Rightarrow \vec{r} = (2-6t)\mathbf{i} + (1+2t)\mathbf{j} + (3-4t)\mathbf{k}$$

$$\Rightarrow \vec{r} = 2(1-3t)\mathbf{i} + (1+2t)\mathbf{j} + (3-4t)\mathbf{k}$$

12) Find the vector equation of the plane passing through the points $\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $-5\mathbf{j} - \mathbf{k}$ and $-3\mathbf{i} + 5\mathbf{j}$.

Sol. Let $\vec{a} = \mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

$$\vec{b} = -5\mathbf{j} - \mathbf{k}$$

$$\vec{c} = -3\mathbf{i} + 5\mathbf{j}$$

$\therefore$ Vector equation of the plane passing through $\vec{a}$, $\vec{b}$ and $\vec{c}$ is,

$$\vec{r} = (1-t-s)\vec{a} + t\vec{b} + s\vec{c}, t, s \in \mathbb{R}$$

$$\Rightarrow \vec{r} = (1-t-s)(\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}) + t(-5\mathbf{j} - \mathbf{k}) + s(-3\mathbf{i} + 5\mathbf{j})$$

13) Find the vector equation of the plane passing through the points $(0, 0, 0)$, $(0, 5, 0)$ and $(2, 0, 1)$.

Sol. $\vec{a} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \vec{0}$

$$\vec{b} = 0\mathbf{i} + 5\mathbf{j} + 0\mathbf{k} = 5\mathbf{j}$$

$$\vec{c} = 2\mathbf{i} + 0\mathbf{j} + 1\mathbf{k} = 2\mathbf{i} + \mathbf{k}$$

Vector equation of the plane passing through $\vec{a}$, $\vec{b}$ and $\vec{c}$ is,

$$\vec{r} = (1-t-s)\vec{a} + t\vec{b} + s\vec{c}, t, s \in \mathbb{R}$$

$$\Rightarrow \vec{r} = (1-t-s)\vec{0} + t(5\mathbf{j}) + s(2\mathbf{i} + \mathbf{k})$$

$$\Rightarrow \vec{r} = (5t)\mathbf{j} + s(2\mathbf{i} + \mathbf{k})$$

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

1) Show that the points $A(2\mathbf{i} - \mathbf{j} + \mathbf{k})$, $B(\mathbf{i} - 3\mathbf{j} - 5\mathbf{k})$, $C(3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k})$ are the vertices of a right angle triangle.

Sol. Let 'O' be the origin, then

$$\vec{OA} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\overrightarrow{OB} = i - 3j - 5k$$

$$\overrightarrow{OC} = 3i - 4j - 4k$$

$$\begin{aligned}\therefore \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} = (i - 3j - 5k) - (2i - j + k) \\ &= (1-2)i + (-3+1)j + (-5-1)k\end{aligned}$$

$$\overrightarrow{AB} = -i - 2j - 6k$$

$$\Rightarrow |\overrightarrow{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (3i - 4j - 4k) - (i - 3j - 5k)$$

$$\overrightarrow{BC} = (3-1)i + (-4+3)j + (-4+5)k = 2i - j + k$$

$$|\overrightarrow{BC}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$$

$$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = (2i - j + k) - (3i - 4j - 4k)$$

$$\overrightarrow{CA} = (2-3)i + (-1+4)j + (1+4)k = -i + 3j + 5k$$

$$|\overrightarrow{CA}| = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{1+9+25} = \sqrt{35}$$

$$|\overrightarrow{AB}|^2 = (\sqrt{41})^2 = (\sqrt{6})^2 + (\sqrt{35})^2$$

$$\Rightarrow |\overrightarrow{AB}|^2 = |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2$$

$\Rightarrow$ A, B, C are the vertices of a right angle triangle.

2) Is the triangle formed by the vectors $3i + 5j + 2k$, $2i - 3j - 5k$ and $-5i - 2j + 3k$ equilateral?

Sol. In $\triangle ABC$, let $\overrightarrow{AB} = 3i + 5j + 2k$

$$\overrightarrow{BC} = 2i - 3j - 5k$$

$$\overrightarrow{CA} = -5i - 2j + 3k$$

$$|\overrightarrow{AB}| = \sqrt{3^2 + 5^2 + 2^2} = \sqrt{9+25+4} = \sqrt{38}$$

$$|\overrightarrow{BC}| = \sqrt{2^2 + (-3)^2 + (-5)^2} = \sqrt{4+9+25} = \sqrt{38}$$

$$|\overrightarrow{CA}| = \sqrt{(-5)^2 + (-2)^2 + 3^2} = \sqrt{25+4+9} = \sqrt{38}$$

$$\therefore |\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{CA}| \Rightarrow \triangle ABC \text{ is an equilateral triangle.}$$

3) $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanar vectors. Prove that the following four points are coplanar.

$$(i) -\vec{a} + 4\vec{b} - 3\vec{c}, 3\vec{a} + 2\vec{b} - 5\vec{c}, -3\vec{a} + 8\vec{b} - 5\vec{c}, -3\vec{a} + 2\vec{b} + \vec{c}$$

$$(ii) 6\vec{a} + 2\vec{b} - \vec{c}, 2\vec{a} - \vec{b} + 3\vec{c}, -\vec{a} + 2\vec{b} - 4\vec{c}, -12\vec{a} - \vec{b} - 3\vec{c}$$

Sol. (i) Let 'O' be the origin. Then the position vectors of A, B, C, D are

$$\overrightarrow{OA} = -\vec{a} + 4\vec{b} - 3\vec{c}$$

$$\overrightarrow{OB} = 3\vec{a} + 2\vec{b} - 5\vec{c}$$

$$\overrightarrow{OC} = -3\vec{a} + 8\vec{b} - 5\vec{c}$$

$$\overrightarrow{OD} = -3\vec{a} + 2\vec{b} + \vec{c}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (3\vec{a} + 2\vec{b} - 5\vec{c}) - (-\vec{a} + 4\vec{b} - 3\vec{c}) = 4\vec{a} - 2\vec{b} - 2\vec{c}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (-3\vec{a} + 8\vec{b} - 5\vec{c}) - (-\vec{a} + 4\vec{b} - 3\vec{c}) = -2\vec{a} + 4\vec{b} - 2\vec{c}$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (-3\vec{a} + 2\vec{b} + \vec{c}) - (-\vec{a} + 4\vec{b} - 3\vec{c}) = -2\vec{a} - 2\vec{b} + 4\vec{c}$$

$$A, B, C, D \text{ are coplanar} \Leftrightarrow \begin{vmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{vmatrix} = 4(16 - 4) + 2(-8 - 4) - 2(4 + 8)$$

$$= 4(12) + 2(-12) - 2(12)$$

$$= 48 - 24 - 24$$

$$= 0$$

$\Rightarrow$ A, B, C, D are coplanar.

Second Method:

A, B, C, D are coplanar $\Leftrightarrow \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar.

$$\Leftrightarrow \overrightarrow{AB} = x\overrightarrow{AC} + y\overrightarrow{AD}$$

where x, y are scalars.

$$\Rightarrow 4\vec{a} - 2\vec{b} - 2\vec{c} = x(-2\vec{a} + 4\vec{b} - 2\vec{c}) + y(-2\vec{a} - 2\vec{b} + 4\vec{c})$$

$$\Rightarrow 4\vec{a} - 2\vec{b} - 2\vec{c} + 2ax - 4bx + 2cx + 2ay + 2by - 4cy = 0$$

$$\Rightarrow (4+2x+2y)\vec{a} + (-2-4x+2y)\vec{b} + (-2+2x-4y)\vec{c} = 0$$

$\therefore \vec{a}, \vec{b}, \vec{c}$ are non-coplanar

$$\Rightarrow 4 + 2x + 2y = 0 \quad \dots\dots\dots(1)$$

$$-2 - 4x + 2y = 0 \quad \dots\dots\dots(2)$$

$$-2 + 2x - 4y = 0 \quad \dots\dots\dots(3)$$

Solving (1) and (2)

$$2x + 2y + 4 = 0$$

$$-4x + 2y - 2 = 0$$

$$\begin{array}{r} + \quad - \quad + \\ \hline 6x \quad + 6 = 0 \end{array}$$

$$x = -6/6 = -1$$

Substituting $x = -1$ in equation (1), we get

$$4 + 2(-1) + 2y = 0$$

$$4 - 2 + 2y = 0$$

$$2 + 2y = 0$$

$$2y = -2$$

$$y = -2 / 2 = -1$$

Substituting $x = -1, y = -1$ in equation (3), we get

$$-2 + 2(-1) - 4(-1) = -2 - 2 + 4 = -4 + 4 = 0$$

$\therefore \overline{AB}, \overline{AC}, \overline{AD}$ are coplanar.

$\Rightarrow$ A, B, C, D are coplanar.

$\therefore$ Given points are coplanar.

(ii) Let 'O' be the origin. Then the position vectors of A, B, C, D are

$$\overline{OA} = 6\overline{a} + 2\overline{b} - \overline{c}$$

$$\overline{OB} = 2\overline{a} - \overline{b} + 3\overline{c}$$

$$\overline{OC} = -\overline{a} + 2\overline{b} - 4\overline{c}$$

$$\overline{OD} = -12\overline{a} - \overline{b} - 3\overline{c} \text{ respectively}$$

$$\overline{AB} = \overline{OB} - \overline{OA} = (2\overline{a} - \overline{b} + 3\overline{c}) - (6\overline{a} + 2\overline{b} - \overline{c}) = -4\overline{a} - 3\overline{b} + 4\overline{c}$$

$$\overline{AC} = \overline{OC} - \overline{OA} = (-\overline{a} + 2\overline{b} - 4\overline{c}) - (6\overline{a} + 2\overline{b} - \overline{c}) = -7\overline{a} - 3\overline{c}$$

$$\overline{AD} = \overline{OD} - \overline{OA} = (-12\overline{a} - \overline{b} - 3\overline{c}) - (6\overline{a} + 2\overline{b} - \overline{c}) = -18\overline{a} - 3\overline{b} - 2\overline{c}$$

$$\text{A, B, C, D are coplanar} \Leftrightarrow \begin{vmatrix} -4 & -3 & 4 \\ -7 & 0 & -3 \\ -18 & -3 & -2 \end{vmatrix} = 0$$

$$\begin{vmatrix} -4 & -3 & 4 \\ -7 & 0 & -3 \\ -18 & -3 & -2 \end{vmatrix} = -4(0 - 9) + 3(14 - 54) + 4(21 - 0)$$

$$= 36 + 3(-40) + 4(21)$$

$$= 36 - 120 + 84$$

$$= 120 - 120 = 0$$

$\Rightarrow$ A, B, C, D are coplanar.

4) If $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the positive direction of the coordinate axes, then show that the four points $4\mathbf{i} + 5\mathbf{j} + \mathbf{k}, -\mathbf{j} - \mathbf{k}, 3\mathbf{i} + 9\mathbf{j} + 4\mathbf{k}$ and $-4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$ are coplanar.

Sol. Let 'O' be the origin and A, B, C, D be the given points.

$$\text{Then, } \overrightarrow{OA} = 4i + 5j + k$$

$$\overrightarrow{OB} = -j - k$$

$$\overrightarrow{OC} = 3i + 9j + 4k$$

$$\overrightarrow{OD} = -4i + 4j + 4k$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-j - k) - (4i + 5j + k) = -4i - 6j - 2k$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (3i + 9j + 4k) - (4i + 5j + k) = -i + 4j + 3k$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (-4i + 4j + 4k) - (4i + 5j + k) = -8i - j + 3k$$

$$A, B, C, D \text{ are coplanar} \Leftrightarrow \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = 0$$

$$\begin{aligned} \therefore \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} &= -4(12 + 3) + 6(-3 + 24) - 2(1 + 32) \\ &= -4(15) + 6(21) - 2(33) \\ &= -60 + 126 - 66 \\ &= -126 + 126 \\ &= 0 \end{aligned}$$

$\Rightarrow$ A, B, C, D are coplanar.

5) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanar vectors, then test for the collinearity of the following points whose position vectors are given by

$$(i) \quad \vec{a} - 2\vec{b} + 3\vec{c}, 2\vec{a} + 3\vec{b} - 4\vec{c}, -7\vec{b} + 10\vec{c}$$

$$(ii) \quad 3\vec{a} - 4\vec{b} + 3\vec{c}, -4\vec{a} + 5\vec{b} - 6\vec{c}, 4\vec{a} - 7\vec{b} + 6\vec{c}$$

Sol. (i) Let 'O' be the origin and A, B, C be the given points.

$$\overrightarrow{OA} = \vec{a} - 2\vec{b} + 3\vec{c}$$

$$\overrightarrow{OB} = 2\vec{a} + 3\vec{b} - 4\vec{c}$$

$$\overrightarrow{OC} = -7\vec{b} + 10\vec{c}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\vec{a} + 3\vec{b} - 4\vec{c}) - (\vec{a} - 2\vec{b} + 3\vec{c}) = \vec{a} + 5\vec{b} - 7\vec{c} \quad \dots(1)$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (-7\vec{b} + 10\vec{c}) - (2\vec{a} + 3\vec{b} - 4\vec{c}) = -2\vec{a} - 10\vec{b} + 14\vec{c}$$

$$\overrightarrow{BC} = -2(\vec{a} + 5\vec{b} - 7\vec{c})$$

$$\overrightarrow{BC} = -2 \overrightarrow{AB} \quad [\because \text{from (1)}]$$

$$\Rightarrow \overline{BC} = 2 \overline{BA}$$

$\Rightarrow$ A, B, C are collinear.

(ii) Let 'O' be the origin A, B, C be the given points

$$\overline{OA} = 3\bar{a} - 4\bar{b} + 3\bar{c}$$

$$\overline{OB} = -4\bar{a} + 5\bar{b} - 6\bar{c}$$

$$\overline{OC} = 4\bar{a} - 7\bar{b} + 6\bar{c}$$

$$\overline{AB} = \overline{OB} - \overline{OA} = (-4\bar{a} + 5\bar{b} - 6\bar{c}) - (3\bar{a} - 4\bar{b} + 3\bar{c}) = -7\bar{a} + 9\bar{b} - 9\bar{c}$$

$$\overline{BC} = \overline{OC} - \overline{OB} = (4\bar{a} - 7\bar{b} + 6\bar{c}) - (-4\bar{a} + 5\bar{b} - 6\bar{c}) = 8\bar{a} - 12\bar{b} + 12\bar{c}$$

$$\overline{AB} \neq \lambda \overline{BC}, \text{ where } \lambda \text{ is a scalar.}$$

$\Rightarrow$ A, B, C are non-collinear.

6) If the points whose position vectors are $3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $-\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $4\mathbf{i} + 5\mathbf{j} + \lambda\mathbf{k}$ are coplanar, then show that $\lambda = \frac{-146}{17}$.

Sol. Let 'O' be the origin and A, B, C, D be given points.

$$\overline{OA} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$$\overline{OB} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$$

$$\overline{OC} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

$$\overline{OD} = 4\mathbf{i} + 5\mathbf{j} + \lambda\mathbf{k} \text{ respectively.}$$

$$\Rightarrow \overline{AB} = \overline{OB} - \overline{OA} = (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) - (3\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = -\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$$

$$\overline{AC} = \overline{OC} - \overline{OA} = (-\mathbf{i} + \mathbf{j} + 2\mathbf{k}) - (3\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = -4\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$

$$\overline{AD} = \overline{OD} - \overline{OA} = (4\mathbf{i} + 5\mathbf{j} + \lambda\mathbf{k}) - (3\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = \mathbf{i} + 7\mathbf{j} + (\lambda + 1)\mathbf{k}$$

$$\text{A, B, C, D are coplanar} \Leftrightarrow \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda + 1 \end{vmatrix} = 0$$

$$-1[3(\lambda + 1) - 21] - 5[-4(\lambda + 1) - 3] - 3[-28 - 3] = 0$$

$$-1(3\lambda + 3 - 21) - 5(-4\lambda - 4 - 3) - 3(-31) = 0$$

$$-1(3\lambda - 18) - 5(-4\lambda - 7) + 93 = 0$$

$$-3\lambda + 18 + 20\lambda + 35 + 93 = 0$$

$$17\lambda + 146 = 0$$

$$17\lambda = -146$$

$$\therefore \lambda = -\frac{146}{17}$$

- 7) Find the vector equation of the plane which passes through the points $2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$, $2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and parallel to the vector $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Also find the point where this plane meets the line joining the points $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

Sol. Let $\vec{a} = 2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$
 $\vec{b} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$
 $\vec{c} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$\therefore$ Vector equation of plane passing through $\vec{a}$, $\vec{b}$ and parallel to $\vec{c}$ is given by,

$$\vec{r} = (1-t)\vec{a} + t\vec{b} + s\vec{c}, \quad t, s \in \mathbb{R}$$

$$\vec{r} = (1-t)(2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) + t(2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) + s(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

$$\vec{r} = (2 - 2t + 2t + 3s)\mathbf{i} + (4 - 4t + 3t - 2s)\mathbf{j} + (2 - 2t + 5t + s)\mathbf{k}$$

$$\vec{r} = (2 + 3s)\mathbf{i} + (4 - t - 2s)\mathbf{j} + (2 + 3t + s)\mathbf{k} \quad \dots\dots\dots (1)$$

Let $\vec{p} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$
 $\vec{q} = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

Vector equation of line passing through $\vec{p}$ and $\vec{q}$ is given by,

$$\vec{r} = (1-x)\vec{p} + x\vec{q}, \quad x \in \mathbb{R}$$

$$\vec{r} = (1-x)(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + x(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$$

$$\vec{r} = (2 - 2x + 4x)\mathbf{i} + (1 - x - 2x)\mathbf{j} + (3 - 3x + 3x)\mathbf{k}$$

$$\vec{r} = (2 + 2x)\mathbf{i} + (1 - 3x)\mathbf{j} + 3\mathbf{k} \quad \dots\dots\dots (2)$$

Equating the corresponding coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ from (1) & (2), we get

$$2 + 3s = 2 + 2x \Rightarrow 2x - 3s = 0. \quad \dots\dots\dots (3)$$

$$4 - t - 2s = 1 - 3x \Rightarrow 3x - 2s - t = -3 \quad \dots\dots\dots (4)$$

$$2 + 3t + s = 3 \Rightarrow s + 3t = 1$$

$$\Rightarrow 3t = 1 - s \Rightarrow t = \frac{1-s}{3}$$

Substituting 't' value in equation (4), we get

$$3x - 2s - \left(\frac{1-s}{3}\right) = -3$$

$$9x - 6s - 1 + s = -9$$

$$\Rightarrow 9x - 5s = -8 \quad \dots\dots\dots (5)$$

Solving (3) & (5), we get

$$(2x - 3s = 0) \times 5 \quad \Rightarrow \quad 10x - 15s = 0$$

$$(9x - 5s = -8) \times -3 \quad \Rightarrow \quad \frac{-27x + 15s = 24}{-17x = 24}$$

$$x = \frac{-24}{17}$$

Substituting $x = \frac{-24}{17}$ in (2), we get

$$\vec{r} = \left(2 + 2\left(\frac{-24}{17}\right)\right)i + \left(1 - 3\left(\frac{-24}{17}\right)\right)j + 3k$$

$$\vec{r} = \left(2 - \frac{48}{17}\right)i + \left(1 + \frac{72}{17}\right)j + 3k$$

$$\vec{r} = \left(\frac{34 - 48}{17}\right)i + \left(\frac{17 + 72}{17}\right)j + 3k$$

$$\Rightarrow \vec{r} = \frac{-14}{17}i + \frac{89}{17}j + 3k$$

$$\therefore \text{Point of intersection of plane and line} = \left(\frac{-14}{17}, \frac{89}{17}, 3\right)$$

- 8) Find the vector equation of the plane passing through points $4i - 3j - k$, $3i + 7j - 10k$ and $2i + 5j - 7k$ show that the point $i + 2j - 3k$ lies in the plane.

Sol. Let $\vec{a} = 4i - 3j - k$
 $\vec{b} = 3i + 7j - 10k$
 $\vec{c} = 2i + 5j - 7k$
 $\vec{d} = i + 2j - 3k$

Vector equation of plane passing through $\vec{a}$, $\vec{b}$ and $\vec{c}$ is

$$\vec{r} = (1 - t - s)\vec{a} + t\vec{b} + s\vec{c} \quad t, s \in \mathbb{R}$$

$$\vec{r} = (1 - t - s)(4i - 3j - k) + t(3i + 7j - 10k) + s(2i + 5j - 7k)$$

If the point $\vec{d}$ lies on this plane, then

$$i + 2j - 3k = (1 - t - s)(4i - 3j - k) + t(3i + 7j - 10k) + s(2i + 5j - 7k)$$

$$i + 2j - 3k = (4 - 4t - 4s + 3t + 2s)i + (-3 + 3t + 3s + 7t + 5s)j + (-1 + t + s - 10t - 7s)k$$

$$i + 2j - 3k = (4 - t - 2s)i + (-3 + 10t + 8s)j + (-1 - 9t - 6s)k$$

Equating the coefficient of i, j, k on both sides, we get

$$4 - t - 2s = 1 \Rightarrow t + 2s = 3 \quad \dots\dots\dots(1)$$

$$-3 + 10t + 8s = 2 \Rightarrow 10t + 8s = 5 \quad \dots\dots\dots(2)$$

$$-1 - 9t - 6s = -3 \Rightarrow 9t + 6s = 2 \quad \dots\dots\dots(3)$$

Solving (1) & (2)

$$(t + 2s = 3) \times -4 \Rightarrow -4t - 8s = -12$$

$$10t + 8s = 5 \Rightarrow 10t + 8s = 5$$

$$6t = -7 \Rightarrow t = \frac{-7}{6}$$

From (1) $t + 2s = 3$

$$\frac{-7}{6} + 2s = 3$$

$$2s = 3 + \frac{7}{6} = \frac{18+7}{6}$$

$$2s = \frac{25}{6} \Rightarrow s = \frac{25}{12}$$

From (3)

$$\text{LHS} = 9t + 6s$$

$$= 9\left(\frac{-7}{6}\right) + 6\left(\frac{25}{12}\right) = \frac{-21}{2} + \frac{25}{2} = \frac{-21+25}{2} = \frac{4}{2} = 2 = \text{R.H.S.}$$

$$\therefore t = \frac{-7}{6}, s = \frac{25}{12} \text{ satisfy (1), (2) and (3) equations.}$$

$\Rightarrow \vec{d}$ lies on the plane passing through $\vec{a}$, $\vec{b}$ and $\vec{c}$.

- 9) Show that the line joining the pair of points $6\vec{a} - 4\vec{b} + 4\vec{c}$, $-4\vec{c}$ and the line joining the pair of points $-\vec{a} - 2\vec{b} - 3\vec{c}$, $\vec{a} + 2\vec{b} - 5\vec{c}$ intersect at the point $-4\vec{c}$ when $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanar vectors.

Sol. Equation of the line joining the first pair of points is,

$$\vec{r} = (1-t)(-4\vec{c}) + t(6\vec{a} - 4\vec{b} + 4\vec{c}), \quad t \in \mathbb{R}$$

$$\vec{r} = (6t)\vec{a} + (-4t)\vec{b} + (-4 + 4t + 4t)\vec{c}$$

$$\vec{r} = (6t)\vec{a} + (-4t)\vec{b} + (8t-4)\vec{c} \quad \dots\dots\dots(1)$$

Equation of the line joining the second pair of points is,

$$\vec{r} = (1-s)(-\vec{a} - 2\vec{b} - 3\vec{c}) + s(\vec{a} + 2\vec{b} - 5\vec{c}), \quad s \in \mathbb{R}$$

$$\vec{r} = (-1+s)\vec{a} + (-2+2s)\vec{b} + (-3+3s-5s)\vec{c}$$

$$\vec{r} = (2s-1)\vec{a} - (4s-2)\vec{b} + (-2s-3)\vec{c} \quad \dots\dots\dots(2)$$

Equating the corresponding coefficients of $\vec{a}$, $\vec{b}$ and $\vec{c}$ in (1) & (2), we have

$$6t = 2s - 1 \quad \Rightarrow \quad 6t - 2s = -1 \quad \dots\dots\dots(3)$$

$$-4t = 4s - 2 \quad \Rightarrow \quad 4t + 4s = 2 \Rightarrow 2t + 2s = 1 \quad \dots\dots\dots(4)$$

$$8t - 4 = -2s - 3 \Rightarrow 8t + 2s = 1 \quad \dots\dots\dots(5)$$

Solving (3) & (4), we get

$$6t - 2s = -1$$

$$\underline{2t + 2s = 1}$$

$$8t = 0 \quad \Rightarrow \quad t = 0$$

$$\begin{aligned}\text{From (4)} \quad 2t + 2s &= 1 \\ 2(0) + 2s &= 1 \\ 2s &= 1 \Rightarrow s = \frac{1}{2}\end{aligned}$$

$t = 0, s = \frac{1}{2}$ satisfy equation (5).

$\therefore$ Substituting the value of $t = 0$ in (1) or $s = \frac{1}{2}$ in (2), the point of intersection of the lines is $-4c$.

10) Find the point of intersection of the line $\vec{r} = 2\vec{a} + \vec{b} + t(\vec{b} - \vec{c})$ and the plane

$\vec{r} = \vec{a} + x(\vec{b} + \vec{c}) + y(\vec{a} + 2\vec{b} - \vec{c})$ where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors.

Sol. Given line is, $\vec{r} = 2\vec{a} + \vec{b} + t(\vec{b} - \vec{c})$ (1)

plane is, $\vec{r} = \vec{a} + x(\vec{b} + \vec{c}) + y(\vec{a} + 2\vec{b} - \vec{c})$ (2)

At the point of intersection of the line and the plane, we have,

$$2\vec{a} + \vec{b} + t(\vec{b} - \vec{c}) = \vec{a} + x(\vec{b} + \vec{c}) + y(\vec{a} + 2\vec{b} - \vec{c})$$

$$2\vec{a} + (1+t)\vec{b} - t\vec{c} = (1+y)\vec{a} + (x+2y)\vec{b} + (x-y)\vec{c}$$

$\therefore$ On comparing the corresponding coefficients,

$$2 = 1 + y \Rightarrow y = 2 - 1 = 1 \Rightarrow y = 1$$

$$1 + t = x + 2y \Rightarrow 1 + t = x + 2(1) \Rightarrow t - x = 1 \quad \text{.....(3)}$$

$$-t = x - y \Rightarrow -t = x - 1 \Rightarrow t + x = 1 \quad \text{..... (4)}$$

Solving (3) & (4)

$$t - x = 1$$

$$t + x = 1$$

$$2t = 2$$

$$t = 1$$

$$\text{From (4)} \quad t + x = 1$$

$$1 + x = 1$$

$$\Rightarrow x = 1 - 1$$

$$\Rightarrow x = 0$$

Substituting $t = 1$ in (1) or substituting $x = 0, y = 1$ in (2), we get the point of intersection of (1) & (2) as $2\vec{a} + 2\vec{b} - \vec{c}$.



Vector Products

Scalar or Dot Product of two vectors

Let $\vec{a}$ and $\vec{b}$ be two vectors. The scalar (or dot) product of $\vec{a}$ and $\vec{b}$ written as $\vec{a} \cdot \vec{b}$ is defined as

$$\vec{a} \cdot \vec{b} = 0 \text{ if one of } \vec{a} \text{ or } \vec{b} \text{ is } 0$$

$$= |\vec{a}| |\vec{b}| \cos \theta \text{ if } \vec{a} \neq 0, \vec{b} \neq 0 \text{ and } \theta \text{ is the angle between } \vec{a} \text{ and } \vec{b}$$

Note:

- (i) $\vec{a} \cdot \vec{b}$ is a scalar
- (ii) If $\vec{a}$, $\vec{b}$ are non zero vectors, then $\vec{a} \cdot \vec{b}$ is positive (or) zero (or) negative according as the angle θ between $\vec{a}$ and $\vec{b}$ is acute or right or obtuse angle.

- (iii) If $\theta = 0^\circ \Rightarrow$

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}|$$

$$\vec{a} \cdot \vec{a} = |\vec{a}| |\vec{a}|$$

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

Orthogonal Projection

Let $\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{CD}$ be two non zero vectors. Let P and Q be the feet of the perpendiculars drawn from C and D respectively onto the line AB. The PQ is called the orthogonal projection vector of $\vec{b}$ on $\vec{a}$ and the magnitude $\overline{PQ}$ Then is called the magnitude of the projection of $\vec{b}$ on $\vec{a}$.

1. The projection vector of $\vec{b}$ on $\vec{a}$ is $\frac{(\vec{b} \cdot \vec{a}) \vec{a}}{|\vec{a}|^2}$ and its magnitude is $\frac{|\vec{b} \cdot \vec{a}|}{|\vec{a}|}$

2. The projection vectors $\vec{a}$ on $\vec{b}$ is = $\frac{(\vec{a} \cdot \vec{b})\vec{b}}{|\vec{b}|^2}$

$$\text{magnitude is} = \frac{|\vec{a} \cdot \vec{b}|}{|\vec{b}|}$$

3. Let $\vec{a}, \vec{b}$ be two vector then

- (i) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (commutative law)
- (ii) $(l\vec{a}) \cdot \vec{b} = \vec{a} \cdot (l\vec{b}) = l(\vec{a} \cdot \vec{b}) = l \in \mathbb{R}$
- (iii) $(l\vec{a}) \cdot (m\vec{b}) = lm(\vec{a} \cdot \vec{b}), = l, m \in \mathbb{R}$
- (iv) $(-\vec{a}) \cdot (\vec{b}) = \vec{a} \cdot (-\vec{b}) = -(\vec{a} \cdot \vec{b})$
- (v) $(-\vec{a}) \cdot (-\vec{b}) = \vec{a} \cdot \vec{b}$

Note: If $\vec{i}, \vec{j}, \vec{k}$ are mutually perpendicular unit of vectors then

$$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$$

$$\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$$

Theorem: $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k} \quad \text{Then}$$

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Note: (i) If θ is the angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ then

$$\theta = \cos^{-1} \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right)$$

$$\theta = \cos^{-1} \left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}} \right)$$

(ii) $\vec{a}, \vec{b}$ are perpendicular to each other if and only if $\Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$

Cross Product of two vectors:

Let $\vec{a}$ and $\vec{b}$ be non-zero non collinear vectors. The cross (or vector) product of $\vec{a}$ and $\vec{b}$ written as $\vec{a} \times \vec{b}$ is defined to be the vectors $(|\vec{a}| |\vec{b}| \sin \theta) \vec{n}$ where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\vec{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $(\vec{a}, \vec{b}, \vec{n})$ is a right handed system.

If one of the vectors $\vec{a}, \vec{b}$ is the null vectors or $\vec{a}, \vec{b}$ are collinear vectors then cross product $\vec{a} \times \vec{b}$ is defined as the null vector $\vec{0}$.

Note:

- (1) If $\vec{a}$, $\vec{b}$ are non-zero and non collinear vectors, Then $\vec{a} \times \vec{b}$ is a vector, perpendicular to the plane determined by $\vec{a}$ and $\vec{b}$, whose magnitude is $|\vec{a}||\vec{b}| \sin\theta$.
- (2) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$
- (3) $(-\vec{a}) \times \vec{b} = \vec{a} \times (-\vec{b}) = -(\vec{a} \times \vec{b}) = \vec{b} \times \vec{a}$
- (4) $(-\vec{a}) \times (-\vec{b}) = \vec{a} \times \vec{b}$
- (5) $(l\vec{a}) \times (\vec{b}) = l(\vec{a} \times \vec{b}) = \vec{a} \times (l\vec{b}), l \in \mathbb{R}$
- (6) $(l\vec{a}) \times (m\vec{b}) = lm(\vec{a} \times \vec{b}), l, m \in \mathbb{R}$
- (7) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- (8) $(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$
- (9) If (i, j, k) is on orthogonal triade, Then
 - (i) $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$
 - (ii) $\vec{i} \times \vec{j} = \vec{k}, \vec{j} \times \vec{k} = \vec{i}, \vec{k} \times \vec{i} = \vec{j}$

Theorem: If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

If $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Theorem: For any two vectors $\vec{a}$ and $\vec{b}$

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$

Theorem: The vector area of ΔABC is

$$= \frac{1}{2}(\vec{AB} \times \vec{AC}) = \frac{1}{2}(\vec{BC} \times \vec{BA}) = \frac{1}{2}(\vec{CA} \times \vec{CB})$$

Theorem: If $\vec{a}$, $\vec{b}$, $\vec{c}$ are the position vectors of the vertices A, B and C of ΔABC , Then the

vector area of ΔABC is $\frac{1}{2}(\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b})$ and its area is $\frac{1}{2}|\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}|$

Theorem:

- (i) The vector area of any plane quadrilateral ABCD in terms of diagonals AC and BD is

$$\frac{1}{2}(\vec{AC} \times \vec{BD})$$

- (ii) The area of quadrilateral is $\frac{1}{2}|\vec{AC} \times \vec{BD}|$

- (iii) The vector area of a parallelogram with $\vec{a}$ and $\vec{b}$ as adjacent sides is $\vec{a} \times \vec{b}$ and the area is $|\vec{a} \times \vec{b}|$,

The unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$$= \pm \frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$$

1. If $\vec{a} = 6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\vec{b} = 2\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$ Then find $\vec{a} \cdot \vec{b}$ and angle between $\vec{a}$, $\vec{b}$.

Sol: $\vec{a} = 6\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\vec{b} = 2\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}$ then,

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 \text{ \& } |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$\vec{a} \cdot \vec{b} = 6(2) + 2(-9) + 3(6) = 12 - 18 + 18 = 12$$

$$|\vec{a}| = \sqrt{36 + 4 + 9} = \sqrt{49} = 7$$

$$|\vec{b}| = \sqrt{2^2 + (-9)^2 + 6^2} = \sqrt{4 + 81 + 36} = \sqrt{121} = 11$$

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{(|\vec{a}| |\vec{b}|)} = \frac{12}{7 \times 11} = \frac{12}{77}$$

$$\theta = \cos^{-1} \left(\frac{12}{77} \right)$$

2. If $\vec{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\vec{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ then show that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular to each other.

$$\vec{a} + \vec{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} = 4\mathbf{i} + \mathbf{j} - \mathbf{k}$$

$$\vec{a} - \vec{b} = (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) - (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = -2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$$

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 4(-2) + 1(3) + (-1)(-5)$$

$$= -8 + 3 + 5$$

$$= 0 \quad [\because \vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}]$$

$$\therefore \vec{a} + \vec{b} \perp \vec{a} - \vec{b}$$

3. If $\vec{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}$, $\vec{b} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ then find the orthogonal projection of $\vec{b}$ on $\vec{a}$ and its magnitude.

Sol: Orthogonal projection of $\vec{b}$ on $\vec{a}$ is $= \frac{(\vec{b} \cdot \vec{a})\vec{a}}{|\vec{a}|^2}$

$$\text{Magnitude} = \frac{|\vec{b} \cdot \vec{a}|}{|\vec{a}|}$$

$$\begin{aligned}\vec{b} \cdot \vec{a} &= (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} - \mathbf{j} - \mathbf{k}) \\ &= 2(1) + (-3)(-1) + 1(-1) = 2 + 3 - 1 = 4\end{aligned}$$

$$|\vec{a}| = \sqrt{(1)^2 + (-1)^2 + (-1)^2} = \sqrt{1+1+1} = \sqrt{3}$$

$$\therefore \text{Orthogonal projection of } \vec{b} \text{ on } \vec{a} = \frac{(\vec{b} \cdot \vec{a})\vec{a}}{|\vec{a}|^2} = \frac{4(\mathbf{i} - \mathbf{j} - \mathbf{k})}{(\sqrt{3})^2} = \frac{4(\mathbf{i} - \mathbf{j} - \mathbf{k})}{3}$$

$$\text{Magnitude} = \frac{|4|}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

4. If the vectors $\lambda\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and $2\lambda\mathbf{i} - \lambda\mathbf{j} - \mathbf{k}$ are perpendicular to each other, find λ .

Sol: $\vec{a}, \vec{b}$ are perpendicular then $\vec{a} \cdot \vec{b} = 0$

$$\therefore (\lambda)(2\lambda) + (-3)(-\lambda) + 5(-1) = 0$$

$$2\lambda^2 + 3\lambda - 5 = 0$$

$$2\lambda^2 + 5\lambda - 2\lambda - 5 = 0$$

$$(2\lambda + 5)(\lambda - 1) = 0$$

$$2\lambda + 5 = 0 \text{ (or) } \lambda - 1 = 0$$

$$\lambda = \frac{-5}{2}; \text{ (or) } \lambda = 1$$

5. Prove that the angle θ between any two diagonals of a cube is given by $\cos\theta = \frac{1}{3}$

Sol: Let

$$OA = \mathbf{i}; OB = \mathbf{j}; OC = \mathbf{k}$$

$$\vec{OF}, \vec{GC} \text{ diagonals}$$

$$\vec{OF} = \vec{OA} + \vec{AD} + \vec{DF}$$

$$= \mathbf{i} + \mathbf{k} + \mathbf{j}$$

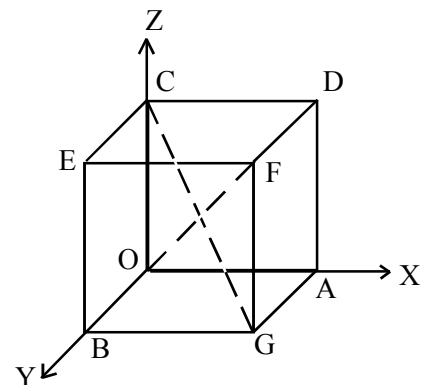
$$= \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\vec{GC} = \vec{GB} + \vec{BO} + \vec{OC}$$

$$= -\mathbf{i} - \mathbf{j} + \mathbf{k}$$

If θ is angle between $\vec{OF}$ and $\vec{GC}$ then

$$\cos\theta = \frac{|\vec{OF} \cdot \vec{GC}|}{|\vec{OF}| |\vec{GC}|} = \frac{|1(-1) + 1(-1) + 1(1)|}{\sqrt{1^2 + 1^2 + 1^2} \sqrt{(-1)^2 + (-1)^2 + 1^2}} = \frac{|1 - 1 + 1|}{\sqrt{3} \cdot \sqrt{3}}$$



6. The Vectors $\overrightarrow{AB} = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{AD} = \mathbf{i} - 2\mathbf{k}$ represent the adjacent sides of a parallelogram ABCD, Find the angle between the diagonals.

Sol: $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$$= 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + \mathbf{i} - 2\mathbf{k}$$

$$= 4\mathbf{i} - 2\mathbf{j}$$

$$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$$

$$= -3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} - \mathbf{i} - 2\mathbf{k}$$

$$= -2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$

If θ is angle between $\overrightarrow{AC}$ and $\overrightarrow{BD}$ then

$$\cos\theta = \frac{\overrightarrow{AC} \cdot \overrightarrow{BD}}{|\overrightarrow{AC}| |\overrightarrow{BD}|} = \frac{4(-2) + (-2)2 + 0(-4)}{\sqrt{4^2 + (-1)^2} \sqrt{(-2)^2 + 2^2 + (-4)^2}}$$

$$\cos\theta = \frac{-8 - 4}{\sqrt{16+4}\sqrt{4+4+16}} = \frac{-12}{\sqrt{20}\sqrt{24}} = \frac{-12}{\sqrt{5 \times 4}\sqrt{6 \times 4}} = \frac{-12}{\sqrt{30}} = \frac{-3}{\sqrt{10} \cdot \sqrt{3}}$$

$$\cos\theta = \frac{-\sqrt{3}}{\sqrt{10}}$$

7. Find the cartesian equation of the plane through A = (2, -1, -4) and parallel to the plane $4x - 12y - 3z - 7 = 0$.

Sol: $4x - 12y - 3z - 7 = 0$ normal to the plane $4\mathbf{i} - 12\mathbf{j} - 3\mathbf{k}$

P = xi + yj + zk be any point on the plan.

$$\overrightarrow{AP} \perp \vec{n}$$

$$(\overrightarrow{OP} - \overrightarrow{OA}) \cdot \vec{n} = 0$$

$$[(x-2)\mathbf{i} + (y+1)\mathbf{j} + (z+4)\mathbf{k}] \cdot (4\mathbf{i} - 12\mathbf{j} - 3\mathbf{k}) = 0$$

$$4(x-2) - 12(y+1) - 3(z+4) = 0$$

$$4x - 12y - 3z - 8 - 12 - 12 = 0$$

$$4x - 12y - 3z - 32 = 0$$

8. Find the angle between the vectors $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

Sol: $\vec{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\vec{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

$$\vec{a} \cdot \vec{b} = 1(3) + 2(-1) + 3(2) = 3 - 2 + 6 = 7$$

$$|\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{9+1+4} = \sqrt{14}$$

If θ is angle between $\vec{a}$ and $\vec{b} \Rightarrow \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$

$$\cos\theta = \frac{7}{\sqrt{14} \cdot \sqrt{14}} = \frac{7}{14} = \frac{1}{2} = \cos 60^\circ$$

$$\theta = 60^\circ$$

- 9.** If the vectors $2\vec{i} + \lambda\vec{j} - \vec{k}$ and $4\vec{i} - 2\vec{j} + 2\vec{k}$ are perpendicular to each other, find λ .

Sol: Let $\vec{a} = 2\vec{i} + \lambda\vec{j} - \vec{k}$; $\vec{b} = 4\vec{i} - 2\vec{j} + 2\vec{k}$

If $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a} \cdot \vec{b} = 0$

$$\therefore 2(4) + \lambda(-2) + (-1)(2) = 0$$

$$8 - 2\lambda - 2 = 0$$

$$2\lambda = 6$$

$$\lambda = 3$$

- 10.** For what value of λ , the vectors $\vec{i} - \lambda\vec{j} + 2\vec{k}$ and $8\vec{i} + 6\vec{j} - \vec{k}$ are at right angles?

Sol: Let $\vec{a} = \vec{i} - \lambda\vec{j} + 2\vec{k}$; $\vec{b} = 8\vec{i} + 6\vec{j} - \vec{k}$

If $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a} \cdot \vec{b} = 0$

$$\therefore 1(8) + (-\lambda)(6) + 2(-1) = 0$$

$$\Rightarrow 8 - 6\lambda - 2 = 0$$

$$\Rightarrow 6\lambda = 6$$

$$\therefore \lambda = 1$$

- 11.** Let $\vec{e}_1$ and $\vec{e}_2$ be unit vectors making angle θ , If $\frac{1}{2}|\vec{e}_1 - \vec{e}_2| = \sin\lambda\theta$, Then find λ .

Sol: $|\vec{e}_1| = 1$; $|\vec{e}_2| = 1$

$$\cos\theta = \frac{\vec{e}_1 \cdot \vec{e}_2}{|\vec{e}_1| |\vec{e}_2|} = \vec{e}_1 \cdot \vec{e}_2$$

$$\frac{1}{2}|\vec{e}_1 - \vec{e}_2| = \sin\lambda\theta$$

$$|\vec{e}_1 - \vec{e}_2| = 2\sin\lambda\theta$$

$$|\vec{e}_1 - \vec{e}_2|^2 = 4\sin^2\lambda\theta$$

$$|\vec{e}_1 - \vec{e}_2| \cdot |\vec{e}_1 - \vec{e}_2| = 4\sin^2\lambda\theta \quad (\because \vec{e}_1 \cdot \vec{e}_1 = |\vec{e}_1|^2)$$

$$|\vec{e}_1|^2 - \vec{e}_1 \cdot \vec{e}_2 - \vec{e}_2 \cdot \vec{e}_1 + |\vec{e}_2|^2 = 4\sin^2\lambda\theta \quad (\because \vec{e}_1 \cdot \vec{e}_2 = \vec{e}_2 \cdot \vec{e}_1)$$

$$1 - 2 \mathbf{e}_1 \cdot \mathbf{e}_2 + 1 = 4\sin^2\lambda\theta$$

$$2 - 2\cos\theta = 4\sin^2\lambda\theta$$

$$\therefore \mathbf{e}_1 \cdot \mathbf{e}_2 = \cos\theta$$

$$2(1 - \cos\theta) = 4\sin^2\lambda\theta$$

$$2(\sin^2\theta/2) = 4\sin^2\lambda\theta$$

$$\therefore 1 - \cos^2\theta = 2\sin^2\theta$$

$$\sin^2\theta/2 = \sin^2\lambda\theta$$

$$\Rightarrow \lambda = \frac{1}{2}$$

- 12.** If $\vec{a} = 2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\vec{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ then find the angle between the vector $2\vec{a} + \vec{b}$ and $\vec{a} + 2\vec{b}$.

Sol: $2\vec{a} + \vec{b} = 2(2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} = 7\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$

$$\vec{a} + 2\vec{b} = 2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + 2(3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 8\mathbf{i} + \mathbf{k}$$

If angle is θ then

$$\cos\theta = \frac{(2\vec{a} + \vec{b}) \cdot (\vec{a} + 2\vec{b})}{|2\vec{a} + \vec{b}| |\vec{a} + 2\vec{b}|} = \frac{7(8) + 3(0) + (-4)(1)}{\sqrt{7^2 + 3^2 + (-4)^2} \cdot \sqrt{8^2 + 1^2}}$$

$$= \frac{56 - 4}{\sqrt{49 + 9 + 16} \cdot \sqrt{64 + 1}} = \frac{52}{\sqrt{74} \cdot \sqrt{65}}$$

$$\theta = \cos^{-1}\left(\frac{52}{\sqrt{74} \cdot \sqrt{65}}\right)$$

- 13.** If $\vec{a} + \vec{b} + \vec{c} = 0$, $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 7$ find the angle between $\vec{a}$ and $\vec{b}$.

Sol: $\vec{a} + \vec{b} = -\vec{c}$

$$(\vec{a} + \vec{b})^2 = (-\vec{c})^2$$

$$(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{c} \cdot \vec{c}$$

$$|\vec{a}|^2 + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + |\vec{b}|^2 = |\vec{c}|^2$$

$$2|\vec{a}||\vec{b}|\cos\theta = 15$$

$$2(3)(5)\cos\theta = 15$$

$$\cos\theta = \frac{1}{2} = \cos 60^\circ$$

$$\theta = 60^\circ$$

- 14.** If $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $|\vec{c}| = 4$ and each of $\vec{a}$, $\vec{b}$, $\vec{c}$ is perpendicular to the sum of the other two vectors, then find the magnitude of $\vec{a} + \vec{b} + \vec{c}$.

Sol: $\bar{a} \perp^r (\bar{b} + \bar{c}) \Rightarrow \bar{a} \cdot (\bar{b} + \bar{c}) = 0$

$$\bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c} = 0$$

$\bar{b} \perp^r (\bar{c} + \bar{a}) \Rightarrow \bar{b} \cdot (\bar{c} + \bar{a}) = 0$

$$\bar{b} \cdot \bar{c} + \bar{b} \cdot \bar{a} = 0$$

$\bar{c} \perp^r (\bar{a} + \bar{b}) \Rightarrow \bar{c} \cdot (\bar{a} + \bar{b}) = 0$

$$\bar{c} \cdot \bar{a} + \bar{c} \cdot \bar{b} = 0$$

$$\bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c} + \bar{b} \cdot \bar{c} + \bar{b} \cdot \bar{a} + \bar{c} \cdot \bar{a} + \bar{c} \cdot \bar{b} = 0$$

$$2(\bar{a} \cdot \bar{b} + \bar{b} \cdot \bar{c} + \bar{c} \cdot \bar{a}) = 0$$

$$|\bar{a} + \bar{b} + \bar{c}|^2 = (\bar{a} + \bar{b} + \bar{c}) \cdot (\bar{a} + \bar{b} + \bar{c})$$

$$= |\bar{a}|^2 + \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c} + \bar{b} \cdot \bar{a} + |\bar{b}|^2 + \bar{b} \cdot \bar{c} + \bar{c} \cdot \bar{a} + \bar{c} \cdot \bar{b} + |\bar{c}|^2$$

$$= |\bar{a}|^2 + |\bar{b}|^2 + |\bar{c}|^2 + 2(\bar{a} \cdot \bar{b} + \bar{b} \cdot \bar{c} + \bar{c} \cdot \bar{a})$$

$$= 2^2 + 3^2 + 4^2 + 2(0)$$

$$= 4 + 9 + 16$$

$$= 29$$

$$\therefore |\bar{a} + \bar{b} + \bar{c}| = \sqrt{29}$$

- 15.** Show that the points (5, -1, 1), (7, -4, 7) (1, -6, 10) and (-1, -3, 4) are the vertices a rhombus.

Sol: $\overrightarrow{OA} = 5\mathbf{i} - \mathbf{j} + \mathbf{k}$

$$\overrightarrow{OB} = 7\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$$

$$\overrightarrow{OC} = \mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\overrightarrow{OD} = -\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$$

$$\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = -8\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -4\mathbf{i} - 5\mathbf{j} + 9\mathbf{k}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$$

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = -2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$$

$$\overrightarrow{DA} = \overrightarrow{OA} - \overrightarrow{OD} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$$

$$|\overrightarrow{AB}| = \sqrt{4+9+36} = 7$$

$$|\overrightarrow{BC}| = \sqrt{36+4+9} = 7$$

$$|\overrightarrow{CD}| = \sqrt{4+9+36} = 7$$

$$|\overrightarrow{DA}| = \sqrt{36+4+9} = 7$$

$$|\overrightarrow{BD}| = \sqrt{64+1+9} = \sqrt{74}$$

$$|\overline{AC}| = \sqrt{16 + 25 + 81} = \sqrt{122}$$

$$\text{i.e., } |\overline{AB}| = |\overline{BC}| = |\overline{CD}| = |\overline{DA}| \text{ \& } BD \neq AC$$

ABCD is a rhombus.

- 16.** If $\overline{a} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, $\overline{b} = -\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ then find $\overline{a} \times \overline{b}$ and unit vector perpendicular to both $\overline{a}$ and $\overline{b}$.

$$\begin{aligned} \text{Sol: } \overline{a} \times \overline{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 5 \\ -1 & 4 & 2 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} -3 & 5 \\ -1 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 5 \\ -1 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} \\ &= \mathbf{i}(-6 - 20) - \mathbf{j}(4 + 5) + \mathbf{k}(8 - 3) \\ &= -26\mathbf{i} - 9\mathbf{j} + 5\mathbf{k} \end{aligned}$$

Unit vector perpendicular to both $\overline{a}$ and $\overline{b}$ is

$$\begin{aligned} &= \pm \frac{\overline{a} \times \overline{b}}{|\overline{a} \times \overline{b}|} \\ &= \pm \frac{(-26\mathbf{i} - 9\mathbf{j} + 5\mathbf{k})}{\sqrt{(-26)^2 + (-9)^2 + 5^2}} = \pm \frac{(-26\mathbf{i} - 9\mathbf{j} + 5\mathbf{k})}{\sqrt{782}} \end{aligned}$$

- 17.** If $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ Then find $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$ and unit vector perpendicular to both $\overline{a} + \overline{b}$ and $\overline{a} - \overline{b}$

$$\text{Sol: } \overline{a} + \overline{b} = \mathbf{i} + \mathbf{j} + 7\mathbf{k}; \quad \overline{a} - \overline{b} = 3\mathbf{i} - 7\mathbf{j} + 3\mathbf{k}$$

$$\begin{aligned} (\overline{a} + \overline{b}) \times (\overline{a} - \overline{b}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 7 \\ 3 & -7 & 3 \end{vmatrix} \\ &= \mathbf{i}(3 + 49) - \mathbf{j}(3 - 21) + \mathbf{k}(-7 - 3) \\ &= 52\mathbf{i} + 18\mathbf{j} - 10\mathbf{k} \end{aligned}$$

$$|(\overline{a} + \overline{b}) \times (\overline{a} - \overline{b})| = \sqrt{(52)^2 + (18)^2 + (-10)^2} = \sqrt{4[(26)^2 + (9)^2 + 5^2]} = 2\sqrt{782}$$

Unit vector perpendicular to $\overline{a} + \overline{b}$, $\overline{a} - \overline{b}$

$$= \pm \frac{(\overline{a} + \overline{b}) \times (\overline{a} - \overline{b})}{|(\overline{a} + \overline{b}) \times (\overline{a} - \overline{b})|}$$

$$\begin{aligned}
 &= \pm \frac{(52i + 18j - 10k)}{\sqrt{(-26)^2 + (-9)^2 + 5^2}} \\
 &= \pm \frac{(26i + 9j - 5k)}{\sqrt{782}}
 \end{aligned}$$

18. Find the area of the parallelogram for which $\vec{a} = 2i - 3j$, $\vec{b} = 3i - k$ are adjacent sides.

Sol: Required area = $|\vec{a} \times \vec{b}|$

$$\begin{aligned}
 \text{Vector area of parallelogram } \vec{a} \times \vec{b} &= \begin{vmatrix} i & j & k \\ 2 & -3 & 0 \\ 3 & 0 & -1 \end{vmatrix} \\
 &= i(3 - 0) - j(-2 - 0) + k(0 + 9) \\
 &= 3i + 2j + 9k
 \end{aligned}$$

$$\begin{aligned}
 \text{Area} &= |\vec{a} \times \vec{b}| = \sqrt{3^2 + 2^2 + 9^2} \\
 &= \sqrt{9 + 4 + 81} \\
 &= \sqrt{94}
 \end{aligned}$$

19. If $\vec{a} = i + 2j + 3k$ and $\vec{b} = 3i + 5j - k$ are two sides of a triangle then find its area.

Sol: Required area $\frac{1}{2} |\vec{a} \times \vec{b}|$

$$\begin{aligned}
 \vec{a} \times \vec{b} &= \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 3 & 5 & -1 \end{vmatrix} \\
 &= i(-2 - 15) - j(-1 - 9) + k(5 - 6) \\
 &= -17i + 10j - k
 \end{aligned}$$

$$\begin{aligned}
 |\vec{a} \times \vec{b}| &= \sqrt{(-17)^2 + (10)^2 + (-1)^2} \\
 &= \sqrt{289 + 100 + 1} \\
 &= \sqrt{390}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{Required area} &= \frac{1}{2} |\vec{a} \times \vec{b}| = \frac{1}{2} |\sqrt{390}| \\
 &= \frac{\sqrt{390}}{2}
 \end{aligned}$$

20. If θ is the angle between $a = 2i - j + k$ and $b = 3i + 4j - k$ then find $\sin\theta$

Sol: $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$$

$$= i(1-4) - j(-2-3) + k(8+3) \\ = -3i + 5j + 11k$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-3)^2 + 5^2 + 11^2} = \sqrt{9+25+121} = \sqrt{155}$$

$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$$

$$|\vec{b}| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{9+16+1} = \sqrt{26}$$

$$\therefore \sin\theta = \frac{\sqrt{155}}{\sqrt{6} \cdot \sqrt{26}} = \frac{\sqrt{155}}{\sqrt{156}}$$

21. Let $a = 2i + j - 2k$, $b = i + j$ If $\vec{c}$ is a vector such that $a \cdot c = |c|$, $|\vec{c} - \vec{a}| = 2\sqrt{2}$ and the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$ is 30° , Then find the value of $|(\vec{a} \times \vec{b}) \times \vec{c}|$.

Sol: $|\vec{a}| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{4+1+4} = 3$

$$|\vec{b}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$|\vec{c} - \vec{a}| = 2\sqrt{2}$$

$$|\vec{c} - \vec{a}|^2 = (2\sqrt{2})^2$$

$$|\vec{c}|^2 + |\vec{a}|^2 - 2(\vec{c} \cdot \vec{a}) = 8$$

$$|\vec{c}|^2 + 9 - 2|\vec{c}| = 8$$

$$|\vec{c}|^2 - 2|\vec{c}| + 1 = 8$$

$$(|\vec{c}| - 1)^2 = 0$$

$$|\vec{c}| = 1$$

$$|(\bar{a} \times \bar{b}) \times \bar{c}| = |\bar{a} \times \bar{b}| |\bar{c}| \sin 30^\circ$$

$$|\bar{a} \times \bar{b}| (1) \frac{1}{2}$$

$$= \frac{1}{2} |\bar{a} \times \bar{b}| \quad \dots\dots\dots(1)$$

$$\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= i(0 + 2) - j(0 + 2) + k(2 - 1)$$

$$= 2i - 2j + k$$

$$|\bar{a} \times \bar{b}| = \sqrt{4 + 4 + 1} = 3$$

$$(1) \Rightarrow |(\bar{a} \times \bar{b}) \times \bar{c}| = \frac{1}{2} (3) = \frac{3}{2}$$

- 22.** Let $\bar{a} = 2i + 5j - k$, $\bar{b} = i - 4j + 5k$ and $\bar{c} = 3i + j - k$, Find vector α which is perpendicular to both $\bar{a}$ and $\bar{b}$ and $\alpha \cdot \bar{c} = 21$.

Sol: There exist λ such that $\bar{\alpha} = \lambda(\bar{a} \times \bar{b})$

$$\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ 4 & 5 & -1 \\ 1 & -4 & 5 \end{vmatrix}$$

$$= i(25 - 4) - j(20 + 1) + k(-16 - 5)$$

$$= 21i - 21j - 21k$$

$$\therefore \bar{\alpha} = \lambda(21i - 21j - 21k)$$

$$= 21\lambda(i - j - k)$$

$$\text{but } \bar{\alpha} \cdot \bar{c} = 21$$

$$21\lambda(i - j - k) \cdot (3i + j - k) = 21$$

$$21\lambda(3 - 1 + 1) = 21$$

$$21 \times 3 \times \lambda = 21$$

$$\lambda = \frac{1}{3}$$

$$\therefore \alpha = 2\left(\frac{1}{3}\right)(i - j - k)$$

$$\alpha = 7(i - j - k) = 7i - 7j - 7k$$

23. For any vector $\vec{a}$ show that $|\vec{a} \times \vec{i}|^2 + |\vec{a} \times \vec{j}|^2 + |\vec{a} \times \vec{k}|^2 = 2|\vec{a}|^2$.

Sol: If $\vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$, then $|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$

$$\begin{aligned}\vec{a} \times \vec{i} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ 1 & 0 & 0 \end{vmatrix} \\ &= \vec{i}(0 - 0) - \vec{j}(0 - z) + \vec{k}(0 - y) \\ &= z\vec{j} - y\vec{k} \\ |\vec{a} \times \vec{i}| &= \sqrt{z^2 + y^2}\end{aligned}$$

Similarly $|\vec{a} \times \vec{j}| = \sqrt{z^2 + x^2}$

$$|\vec{a} \times \vec{k}| = \sqrt{x^2 + y^2}$$

$$\begin{aligned}\therefore |\vec{a} \times \vec{i}|^2 + |\vec{a} \times \vec{j}|^2 + |\vec{a} \times \vec{k}|^2 &= z^2 + y^2 + z^2 + x^2 + x^2 + y^2 = 2(x^2 + y^2 + z^2) \\ &= 2(\sqrt{x^2 + y^2 + z^2})^2 \\ &= 2|\vec{a}|^2\end{aligned}$$

24. If $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$, $\vec{b} = \vec{i} - 3\vec{j} - 5\vec{k}$ then find $|\vec{a} \times \vec{b}|$.

$$\begin{aligned}\text{Sol: } \vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & -3 & -5 \end{vmatrix} \\ &= \vec{i}(5 + 3) - \vec{j}(-10 - 1) + \vec{k}(-6 + 1) \\ &= 8\vec{i} + 11\vec{j} - 5\vec{k} \\ \therefore |\vec{a} \times \vec{b}| &= \sqrt{8^2 + 11^2 + (-5)^2} \\ &= \sqrt{64 + 121 + 25} \\ &= \sqrt{210}\end{aligned}$$

25. If $\vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}$, $\vec{b} = \vec{i} + 4\vec{j} - 2\vec{k}$ then find $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$.

$$\begin{aligned}\text{Sol: } \vec{a} + \vec{b} &= 3\vec{i} + \vec{j} - \vec{k} \\ \vec{a} - \vec{b} &= \vec{i} - 7\vec{j} + 3\vec{k}\end{aligned}$$

$$\begin{aligned}
 (\bar{a} + \bar{b}) \times (\bar{a} - \bar{b}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -1 \\ 1 & -7 & 3 \end{vmatrix} \\
 &= \mathbf{i}(3 - 7) - \mathbf{j}(9 + 1) + \mathbf{k}(-21 - 1) \\
 &= -4\mathbf{i} - 10\mathbf{j} - 22\mathbf{k}
 \end{aligned}$$

26. If $4\mathbf{i} + \frac{2p}{3}\mathbf{j} + p\mathbf{k}$ is parallel to the vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ find p .

Sol: If $\bar{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\bar{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ Parallel then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

$$\therefore \frac{4}{1} = \frac{2p/3}{2} = \frac{p}{3}$$

$$4 = \frac{p}{3}$$

$$\Rightarrow p = 12$$

27. Find unit vector perpendicular to both $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

Sol: The unit vector perpendicular to both $\bar{a}$ and $\bar{b}$ is $= \pm \frac{(\bar{a} \times \bar{b})}{|\bar{a} \times \bar{b}|}$

$$\begin{aligned}
 \bar{a} \times \bar{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix} \\
 &= \mathbf{i}(3 - 1) - \mathbf{j}(3 - 2) + \mathbf{k}(1 - 2) \\
 &= 2\mathbf{i} - \mathbf{j} - \mathbf{k}
 \end{aligned}$$

$$\begin{aligned}
 \therefore |\bar{a} \times \bar{b}| &= \sqrt{2^2 + (-1)^2 + (-1)^2} \\
 &= \sqrt{4 + 1 + 1} = \sqrt{6}
 \end{aligned}$$

$$\therefore \text{Required unit vector} = \pm \frac{(2\mathbf{i} - \mathbf{j} - \mathbf{k})}{\sqrt{6}}$$

28. Find the area of the parallelogram having $\bar{a} = 2\mathbf{j} - \mathbf{k}$ and $\bar{b} = -\mathbf{i} + \mathbf{k}$ as adjacent sides.

Sol: Required area $= |\bar{a} \times \bar{b}|$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -1 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= i(2 - 0) - j(0 - 1) + k(0 + 2)$$

$$= 2i + j + 2k$$

$$\begin{aligned} \therefore |\vec{a} \times \vec{b}| &= \sqrt{2^2 + 1^2 + 2^2} \\ &= \sqrt{4 + 1 + 4} = 3 \end{aligned}$$

29. Find the area of the triangle whose vertices are A(1, 2, 3), B(2, 3, 1) and C(3, 1, 2).

Sol: $\vec{OA} = i + 2j + 3k$

$$\vec{OB} = 2i + 3j + k$$

$$\vec{OC} = 3i + j + 2k$$

$$\vec{AB} = \vec{OB} - \vec{OA} = i + j - 2k$$

$$\vec{AC} = \vec{OC} - \vec{OA} = 2i - j - k$$

$$\begin{aligned} \vec{AB} \times \vec{AC} &= \begin{vmatrix} i & j & k \\ 1 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix} \\ &= i(-1 - 2) - j(-1 + 4) + k(-1 - 2) \\ &= -3i - 3j - 3k \end{aligned}$$

$$|\vec{AB} \times \vec{AC}| = \sqrt{9 + 9 + 9} = \sqrt{27} = 3\sqrt{3}$$

$$\begin{aligned} \text{Required area} &= \frac{1}{2} |\vec{AB} \times \vec{AC}| \\ &= \frac{1}{2} (3\sqrt{3}) \\ &= \frac{3\sqrt{3}}{2} \end{aligned}$$

30. If $\vec{a} = 2i + j - k$, $\vec{b} = -i + 2j - 4k$, $\vec{c} = i + j + k$ then find $(\vec{a} \times \vec{b}) \cdot (\vec{b} \times \vec{c})$.

Sol: $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ -1 & 2 & -4 \end{vmatrix}$

$$\begin{aligned} &= i(-4 + 2) - j(-8 - 1) + k(4 + 1) \\ &= -2i + 9j + 5k \end{aligned}$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} i & j & k \\ -1 & 2 & -4 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\begin{aligned}
&= i(2+4) - j(-1+4) + k(-1-2) \\
&= 6i - 3j - 3k \\
(\bar{a} \times \bar{b}) \cdot (\bar{b} \times \bar{c}) &= (-2i + 9j + 5k) \cdot (6i - 3j - 3k) \\
&= (-2)(6) + (9)(-3) + (5)(-3) \\
&= -12 - 27 - 15 \\
&= -54
\end{aligned}$$

- 31.** Find a unit vector perpendicular to the plane determined by the points P(1, -1, 2), Q(2, 0, -1) and R(0, 2, 1).

Sol: $OP = i - j + 2k$; $OQ = 2i - k$; $OR = 2j + k$

$$PQ = OQ - OP = i + j - 3k$$

$$PR = OR - OP = -i + 3j - k$$

$$\begin{aligned}
PQ \times PR &= \begin{vmatrix} i & j & k \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} \\
&= i(-1+9) - j(-1-3) + k(3+1) \\
&= 8i + 4j + 4k \\
&= 4(2i + j + k)
\end{aligned}$$

$$|PQ \times PR| = 4\sqrt{4+1+1} = 4\sqrt{6}$$

$$\begin{aligned}
\text{Required unit vector} &= \pm \frac{(PQ \times PR)}{|PQ \times PR|} \\
&= \pm \frac{4(2i + j + k)}{4\sqrt{6}} \\
&= \pm \frac{(2i + j + k)}{\sqrt{6}}
\end{aligned}$$

- 32.** If $|\bar{a}| = 13$, $|\bar{b}| = 5$, $\bar{a} \cdot \bar{b} = 60$ then find $|\bar{a} \times \bar{b}|$.

$$\begin{aligned}
\text{Sol: } |\bar{a} \times \bar{b}|^2 &= |\bar{a}|^2 |\bar{b}|^2 - (\bar{a} \cdot \bar{b})^2 \\
&= (13)^2 (5)^2 - (60)^2 \\
&= 4225 - 3600 = 625 \\
\Rightarrow |\bar{a} \times \bar{b}| &= 25
\end{aligned}$$

- 33.** If $\bar{a} = 2i + 3j + 4k$, $\bar{b} = i + j - k$, $\bar{c} = i - j + k$ then compute $\bar{a} \times (\bar{b} \times \bar{c})$ and verify that it is perpendicular to $\bar{a}$.

$$\begin{aligned} \text{Sol: } \bar{\mathbf{b}} \times \bar{\mathbf{c}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \\ &= \mathbf{i}(1 \cdot 1 - (-1) \cdot 1) - \mathbf{j}(1 \cdot 1 - (-1) \cdot 1) + \mathbf{k}(-1 \cdot 1 - 1 \cdot 1) \\ &= -2\mathbf{j} - 2\mathbf{k} \end{aligned}$$

$$\begin{aligned} \bar{\mathbf{a}} \times (\bar{\mathbf{b}} \times \bar{\mathbf{c}}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 0 & -2 & -2 \end{vmatrix} \\ &= \mathbf{i}(-6 + 8) - \mathbf{j}(-4 - 0) + \mathbf{k}(-4 - 0) \\ &= 2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k} \end{aligned}$$

$$\begin{aligned} (\bar{\mathbf{a}} \times (\bar{\mathbf{b}} \times \bar{\mathbf{c}})) \cdot \bar{\mathbf{a}} &= (2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}) \cdot (2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}) \\ &= 2(2) + 4(3) + (-4)(4) \\ &= 4 + 12 - 16 \\ &= 0 \end{aligned}$$

$\therefore \bar{\mathbf{a}} \times (\bar{\mathbf{b}} \times \bar{\mathbf{c}})$ is perpendicular to $\bar{\mathbf{a}}$.

- 34.** If $\bar{\mathbf{a}} = 7\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\bar{\mathbf{b}} = 2\mathbf{i} + 8\mathbf{k}$ and $\bar{\mathbf{c}} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ then compute $\bar{\mathbf{a}} \times \bar{\mathbf{b}}$, $\bar{\mathbf{a}} \times \bar{\mathbf{c}}$, $\bar{\mathbf{a}} \times (\bar{\mathbf{b}} + \bar{\mathbf{c}})$. Verify whether the cross product is distributive over the vector addition.

$$\begin{aligned} \text{Sol: } \bar{\mathbf{a}} \times \bar{\mathbf{b}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -2 & 3 \\ 2 & 0 & 8 \end{vmatrix} \\ &= \mathbf{i}(-16 - 0) - \mathbf{j}(56 - 6) + \mathbf{k}(0 + 4) \\ &= -16\mathbf{i} - 50\mathbf{j} + 4\mathbf{k} \end{aligned}$$

$$\begin{aligned} \bar{\mathbf{a}} \times \bar{\mathbf{c}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -2 & 3 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \mathbf{i}(-2 - 3) - \mathbf{j}(7 - 3) + \mathbf{k}(7 + 2) \\ &= -5\mathbf{i} - 4\mathbf{j} + 9\mathbf{k} \end{aligned}$$

$$\begin{aligned} \bar{\mathbf{b}} + \bar{\mathbf{c}} &= 2\mathbf{i} + 8\mathbf{k} + \mathbf{i} + \mathbf{j} + \mathbf{k} \\ &= 3\mathbf{i} + \mathbf{j} + 9\mathbf{k} \end{aligned}$$

$$\begin{aligned}\bar{a} \times (\bar{b} + \bar{c}) &= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 7 & -2 & 3 \\ 3 & 1 & 9 \end{vmatrix} \\ &= \bar{i}(-18 - 3) - \bar{j}(63 - 9) + \bar{k}(7 + 6) \\ &= -21\bar{i} - 54\bar{j} + 13\bar{k} \quad \dots\dots\dots(1)\end{aligned}$$

$$\begin{aligned}\bar{a} \times \bar{b} + \bar{a} \times \bar{c} &= -16\bar{i} - 50\bar{j} + 4\bar{k} + (-5\bar{i} - 4\bar{j} + 9\bar{k}) \\ &= -21\bar{i} - 54\bar{j} + 13\bar{k} \quad \dots\dots\dots(2)\end{aligned}$$

From (1), (2)..

$$\bar{a} \times (\bar{b} + \bar{c}) = (\bar{a} \times \bar{b}) + (\bar{a} \times \bar{c})$$

$\therefore$ Cross product is distributive over the vector addition.

35. If $\bar{a} = \bar{i} + \bar{j} + \bar{k}$, $\bar{c} = \bar{j} - \bar{k}$ Then find vector $\bar{b}$ such that $\bar{a} \times \bar{b} = \bar{c}$ and $\bar{a} \cdot \bar{b} = 3$.

Sol: Let $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

$$\bar{a} \times \bar{b} = \bar{c}$$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 1 & 1 \\ b_1 & b_2 & b_3 \end{vmatrix} = \bar{j} - \bar{k}$$

$$\bar{i}(b_3 - b_2) - \bar{j}(b_3 - b_1) + \bar{k}(b_2 - b_1) = \bar{j} - \bar{k}$$

$$\Rightarrow b_3 - b_2 = 0; b_1 - b_3 = 1; b_2 - b_1 = -1$$

$$\text{Let } b_3 = b_2 = k$$

$$b_1 - k = 1 \quad \quad \quad k - b_1 = -1$$

$$b_1 = 1 + k; \quad \quad \quad b_1 = k + 1$$

$$\bar{a} \cdot \bar{b} = 3$$

$$(\bar{i} + \bar{j} + \bar{k}) \cdot (b_1\bar{i} + b_2\bar{j} + b_3\bar{k}) = 3$$

$$b_1 + b_2 + b_3$$

$$k + 1 + k + k = 3$$

$$3k = 2$$

$$k = \frac{2}{3}$$

$$\therefore b_1 = \frac{2}{3} + 1 = \frac{5}{3}$$

$$\therefore \bar{b} = \frac{5}{3}\bar{i} + \frac{2}{3}\bar{j} + \frac{2}{3}\bar{k} = \frac{1}{3}(5\bar{i} + 2\bar{j} + 2\bar{k})$$

- 36.** If $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors such that $\vec{a}$ is perpendicular to the plane of $\vec{b}$, $\vec{c}$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{3}$ then find $|\vec{a} + \vec{b} + \vec{c}|$.

Sol: $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

$$\vec{a} \perp \vec{b} \Rightarrow \vec{a} \cdot \vec{b} = 0$$

$$\vec{a} \perp \vec{c} \Rightarrow \vec{a} \cdot \vec{c} = 0$$

$$|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$$

$$= 1 + 1 + 1 + 2\left(0 + |\vec{b}||\vec{c}| \cos \frac{\pi}{3} + 0\right)$$

$$= 1 + 1 + 1 + 2\left(1 \cdot 1 \cdot \frac{1}{2}\right)$$

$$= 1 + 1 + 1 + 1$$

$$= 4$$

$$\therefore |\vec{a} + \vec{b} + \vec{c}| = 2$$

- 37.** $\vec{a} = 3\vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = -\vec{i} + 3\vec{j} + 2\vec{k}$, $\vec{c} = 4\vec{i} + 5\vec{j} - 2\vec{k}$, $\vec{d} = \vec{i} + 3\vec{j} + 5\vec{k}$, then compute

(i) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ (ii) $(\vec{a} \times \vec{b}) \cdot \vec{c} - (\vec{a} \times \vec{d}) \cdot \vec{b}$

Sol: $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ -1 & 3 & 2 \end{vmatrix}$

$$= \vec{i}(-2 - 6) - \vec{j}(6 + 2) + \vec{k}(9 - 1)$$

$$= -8\vec{i} - 8\vec{j} + 8\vec{k}$$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 5 & -2 \\ 1 & 3 & 5 \end{vmatrix}$$

$$= \vec{i}(25 + 6) - \vec{j}(20 + 2) + \vec{k}(12 - 5)$$

$$= 31\vec{i} - 22\vec{j} + 7\vec{k}$$

(i) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -8 & -8 & 8 \\ 31 & -22 & 7 \end{vmatrix}$

$$= \vec{i}(-56 + 176) - \vec{j}(-56 - 248) + \vec{k}(176 + 248)$$

$$= 120\vec{i} + 304\vec{j} + 424\vec{k}$$

$$\begin{aligned}
 (\bar{a} \times \bar{b}) \cdot \bar{c} &= (-8i - 8j + 8k) \cdot (4i + 5j - 2k) \\
 &= (-8)(4) + (-8)(5) + (8)(-2) \\
 &= -32 - 40 - 16 \\
 &= -88
 \end{aligned}$$

$$\begin{aligned}
 (\bar{a} \times \bar{d}) &= \begin{vmatrix} i & j & k \\ -8 & -8 & 8 \\ 31 & -22 & 7 \end{vmatrix} \\
 &= i(-5 - 6) - j(15 - 2) + k(9 + 1) \\
 &= -11i - 13j + 10k
 \end{aligned}$$

$$\begin{aligned}
 (\bar{a} \times \bar{d}) \cdot \bar{b} &= (-11i - 13j + 10k) \cdot (-i + 3j + 21k) \\
 &= (-11)(-1) + (-13)(3) + 10(2) \\
 &= 11 - 39 + 20 \\
 &= -8
 \end{aligned}$$

$$\begin{aligned}
 \therefore (\bar{a} \times \bar{b}) \cdot \bar{c} - (\bar{a} \times \bar{d}) \cdot \bar{b} &= -88 - (-8) \\
 &= -88 + 8 \\
 &= -80
 \end{aligned}$$

38. $\bar{a} = (1, -1, -6)$, $\bar{b} = (1, -3, 4)$, $\bar{c} = (2, -5, 3)$ then compute (i) $\bar{a} \cdot (\bar{b} \times \bar{c})$ (ii) $\bar{a} \times (\bar{b} \times \bar{c})$ (iii) $(\bar{a} \times \bar{b}) \times \bar{c}$

$$\begin{aligned}
 \text{Sol: } \bar{b} \times \bar{c} &= \begin{vmatrix} i & j & k \\ 1 & -3 & 4 \\ 2 & -5 & 3 \end{vmatrix} \\
 &= i(-9 + 20) - j(3 - 8) + k(-5 + 6) \\
 &= 11i + 5j + k
 \end{aligned}$$

$$\begin{aligned}
 \bar{a} \cdot (\bar{b} \times \bar{c}) &= (i - j - 6k) \cdot (11i + 5j + k) \\
 &= 1(11) + (-1)(5) + (-6)(1) \\
 &= 11 - 5 - 6 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \bar{a} \times (\bar{b} \times \bar{c}) &= \begin{vmatrix} i & j & k \\ 1 & -1 & -6 \\ 11 & 5 & 1 \end{vmatrix} \\
 &= i(-1 + 30) - j(1 + 66) + k(5 + 11) \\
 &= 29i - 67j + 16k
 \end{aligned}$$

$$\begin{aligned}\bar{\mathbf{a}} \times \bar{\mathbf{b}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -6 \\ 1 & -3 & 4 \end{vmatrix} \\ &= \mathbf{i}(-4 - 18) - \mathbf{j}(4 + 6) + \mathbf{k}(-3 + 1) \\ &= -22\mathbf{i} - 10\mathbf{j} - 2\mathbf{k}\end{aligned}$$

$$\begin{aligned}(\bar{\mathbf{a}} \times \bar{\mathbf{b}}) \times \bar{\mathbf{c}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -22 & -10 & -2 \\ 2 & -5 & 3 \end{vmatrix} \\ &= \mathbf{i}(-30 - 10) - \mathbf{j}(-66 + 4) + \mathbf{k}(110 + 24) \\ &= -40\mathbf{i} + 62\mathbf{j} + 130\mathbf{k}\end{aligned}$$

39. If $\bar{\mathbf{a}} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\bar{\mathbf{b}} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\bar{\mathbf{c}} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ then compute $\bar{\mathbf{a}} \times (\bar{\mathbf{b}} \times \bar{\mathbf{c}})$, $|(\bar{\mathbf{a}} \times \bar{\mathbf{b}}) \times \bar{\mathbf{c}}|$.

Sol: $\bar{\mathbf{b}} \times \bar{\mathbf{c}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$

$$\begin{aligned}&= \mathbf{i}(-1 - 2) - \mathbf{j}(-3 - 1) + \mathbf{k}(4 - 1) \\ &= -3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}\end{aligned}$$

$$\begin{aligned}\bar{\mathbf{a}} \times (\bar{\mathbf{b}} \times \bar{\mathbf{c}}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ -3 & 3 & 3 \end{vmatrix} \\ &= \mathbf{i}(-6 - 3) - \mathbf{j}(3 + 3) + \mathbf{k}(3 - 6) \\ &= -9\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}\end{aligned}$$

$$\begin{aligned}\bar{\mathbf{a}} \times \bar{\mathbf{b}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} \\ &= \mathbf{i}(-2 - 1) - \mathbf{j}(1 - 2) + \mathbf{k}(1 + 4) \\ &= -3\mathbf{i} + \mathbf{j} + 5\mathbf{k}\end{aligned}$$

$$\begin{aligned}(\bar{\mathbf{a}} \times \bar{\mathbf{b}}) \times \bar{\mathbf{c}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & 5 \\ 1 & 2 & -1 \end{vmatrix} \\ &= \mathbf{i}(-1 - 10) - \mathbf{j}(3 - 5) + \mathbf{k}(-6 - 1) \\ &= -11\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}\end{aligned}$$

$$\begin{aligned}
 |(\bar{a} \times \bar{b}) \times \bar{c}| &= \sqrt{(-11)^2 + 2^2 + (-7)^2} \\
 &= \sqrt{121 + 4 + 49} \\
 &= \sqrt{174}
 \end{aligned}$$

40. If $\bar{a} = i - 2j - 3k$, $\bar{b} = 2i + j - k$, $\bar{c} = i + 3j - 2k$ then verify $\bar{a} \times (\bar{b} \times \bar{c}) \neq (\bar{a} \times \bar{b}) \times \bar{c}$.

Sol: $\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{vmatrix}$

$$\begin{aligned}
 &= i(2 + 3) - j(-1 + 6) + k(1 + 4) \\
 &= 5i - 5j + 5k
 \end{aligned}$$

$$(\bar{a} \times \bar{b}) \times \bar{c} = \begin{vmatrix} i & j & k \\ 5 & -5 & 5 \\ 1 & 3 & -2 \end{vmatrix}$$

$$\begin{aligned}
 &= i(10 - 15) - j(-10 - 5) + k(15 + 5) \\
 &= -5i + 15j + 20k \quad \dots\dots\dots(1)
 \end{aligned}$$

$$\begin{aligned}
 \bar{b} \times \bar{c} &= \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & 3 & -2 \end{vmatrix} \\
 &= i(-2 + 3) - j(-4 + 1) + k(6 - 1) \\
 &= i + 3j + 5k
 \end{aligned}$$

$$\begin{aligned}
 \bar{a} \times (\bar{b} \times \bar{c}) &= \begin{vmatrix} i & j & k \\ 1 & -2 & -3 \\ 1 & 3 & 5 \end{vmatrix} \\
 &= i(-10 + 9) - j(5 + 3) + k(3 + 2) \\
 &= -i - 8j + 5k \quad \dots\dots\dots(2)
 \end{aligned}$$

From (1) and (2)

$$\bar{a} \times (\bar{b} \times \bar{c}) \neq (\bar{a} \times \bar{b}) \times \bar{c}$$



Trigonometric Ratios upto Transformation

1. In a right angled triangle ABC, θ is an acute angle. x is opposite side, y is an adjacent side, z is hypotenuse, then

$$\sin \theta = \frac{x}{z}$$

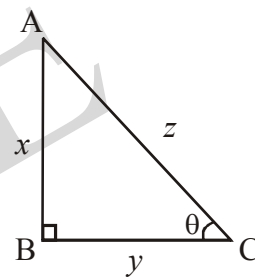
$$\cos \theta = \frac{y}{z}$$

$$\tan \theta = \frac{x}{y}$$

$$\operatorname{cosec} \theta = \frac{z}{x}$$

$$\sec \theta = \frac{z}{y}$$

$$\cot \theta = \frac{y}{x}$$



- * From the definitions of trigonometric ratios, we can observe the following

$$1) \tan \theta = \frac{\sin \theta}{\cos \theta} \quad 2) \cot \theta = \frac{\cos \theta}{\sin \theta}, \quad 3) \sec \theta = \frac{1}{\cos \theta} \quad 4) \cos \theta = \frac{1}{\sec \theta}$$

$$5) \sin \theta = \frac{1}{\operatorname{cosec} \theta} \quad 6) \operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

Trigonometric Identities

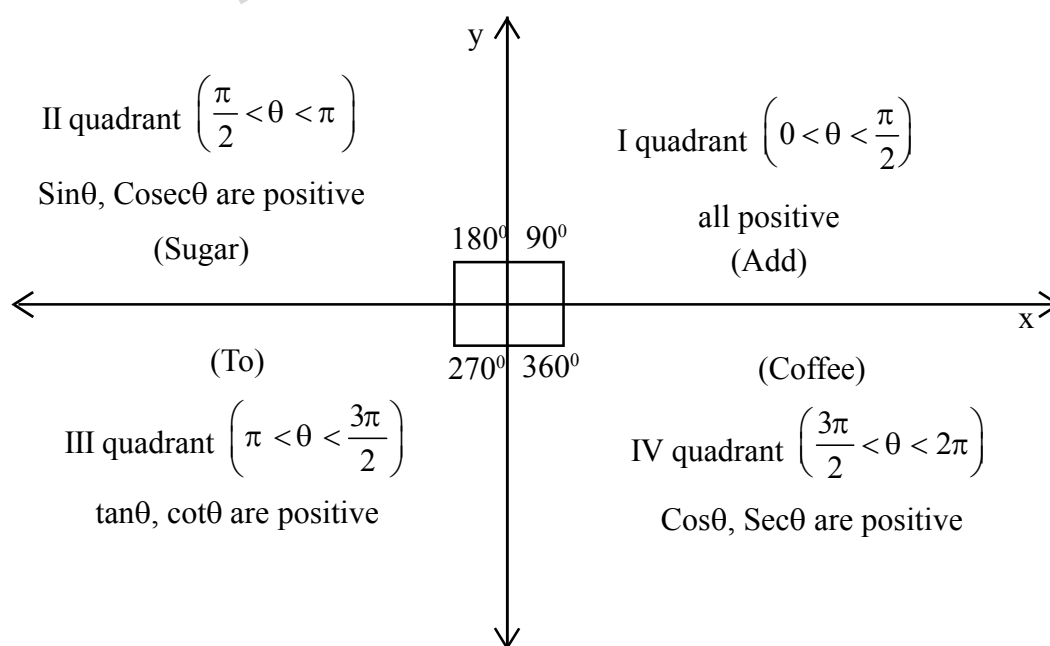
- 1) $\cos^2 \theta + \sin^2 \theta = 1$
 $\cos^2 \theta = 1 - \sin^2 \theta$
 $\sin^2 \theta = 1 - \cos^2 \theta$
- 2) $\sec^2 \theta - \tan^2 \theta = 1$
 $\sec^2 \theta = 1 + \tan^2 \theta$
 $\tan^2 \theta = \sec^2 \theta - 1$

$$\begin{aligned}
 3) \quad & \text{Cosec}^2\theta - \text{Cot}^2\theta = 1 \\
 & \text{Cosec}^2\theta = 1 + \text{Cot}^2\theta \\
 & \text{Cot}^2\theta = \text{Cosec}^2\theta - 1
 \end{aligned}$$

Values of the Trigonometric Functions

Angle (θ)	0°	$\frac{\pi}{6} = 30^\circ$	$\frac{\pi}{4} = 45^\circ$	$\frac{\pi}{3} = 60^\circ$	$\frac{\pi}{2} = 90^\circ$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
$\cot\theta$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\text{cosec}\theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

* We can remember the sign of trigonometric functions in four quadrants by using the following figure.



Add	Sugar	To	Coffee
All Trigonometric functions are +ve	$\left. \begin{matrix} \text{Sin} \\ \text{Cosec} \end{matrix} \right\} +ve$	$\left. \begin{matrix} \tan \\ \text{Cot} \end{matrix} \right\} +ve$	$\left. \begin{matrix} \text{Cos} \\ \text{Sec} \end{matrix} \right\} +ve$
	$\left. \begin{matrix} \text{Cos} \\ \tan \\ \text{Cot} \\ \text{Sec} \end{matrix} \right\} -ve$	$\left. \begin{matrix} \text{Sin} \\ \text{Cos} \\ \text{Sec} \\ \text{Cosec} \end{matrix} \right\} -ve$	$\left. \begin{matrix} \text{Sin} \\ \text{Cosec} \\ \tan \\ \text{Cot} \end{matrix} \right\} -ve$

Angle (α)	$\text{Sin}\alpha$	$\text{Cos}\alpha$	$\tan\alpha$
$n\pi - \theta$	$(-1)^{n+1}\text{Sin}\theta$	$(-1)^n\text{Cos}\theta$	$-\tan\theta$
$n\pi + \theta$	$(-1)^n\text{Sin}\theta$	$(-1)^n\text{Cos}\theta$	$\tan\theta$
$(2n+1)\frac{\pi}{2} - \theta$	$(-1)^n\text{Cos}\theta$	$(-1)^n\text{Sin}\theta$	$\text{Cot}\theta$
$(2n+1)\frac{\pi}{2} + \theta$	$(-1)^n\text{Cos}\theta$	$(-1)^{n+1}\text{Sin}\theta$	$-\text{Cot}\theta$

* Any trigonometric function for the angle $\frac{n\pi}{2} \pm \theta$ ($n \in \mathbb{Z}$),

- (i) If 'n' is even integer, then there is no change in trigonometric function.
- (ii) If 'n' is odd integer, then the change in trigonometric function as follows

$$\text{Sin} \rightleftharpoons \text{Cos} \quad \tan \rightleftharpoons \text{Cot} \quad \text{Sec} \rightleftharpoons \text{Cosec}$$

* $\text{Sin}(-\theta) = -\text{Sin}\theta$, $\text{Cos}(-\theta) = \text{Cos}\theta$; $\tan(-\theta) = -\tan\theta$

$$\text{Cot}(-\theta) = -\text{Cot}\theta, \text{Sec}(-\theta) = \text{Sec}\theta; \text{Cosec}(-\theta) = -\text{Cosec}\theta$$

* All trigonometric functions are periodic functions.

Period of $\text{Sin}x$ is 2π

Period of $\text{Cos}x$ is 2π

Period of $\tan x$ is π

* Range of $\text{Sin}\theta$ (or) $\text{Cos}\theta$ is $[-1, 1]$

Range of $\tan\theta$ (or) $\text{Cot}\theta$ is $\mathbb{R}$

Range of $\text{Sec}\theta$ (or) $\text{Cosec}\theta$ is $[-\infty, -1] \cup [1, \infty]$

Compound Angles

* **A, B are any two angles, then**

i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

iii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

iv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$

* $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

* $\cot(A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$

$$\cot(A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

* If A, B, C $\in$ R then

$$\sin(A+B+C) = \sum (\sin A \cos B \cos C) - \sin A \sin B \sin C$$

$$\cos(A + B + C) = \cos A \cos B \cos C - \sum (\cos A \sin B \sin C)$$

Multiple, sub multiple angles

1. $\sin 2\theta = 2 \sin \theta \cos \theta, \quad \sin \theta = 2 \sin \theta/2 \cos \theta/2$

$$= \frac{2 \tan \theta}{1 + \tan^2 \theta} \quad = \frac{2 \tan \theta/2}{1 + \tan^2 \theta/2}$$

2. $\cos 2\theta = \cos^2 \theta - \sin^2 \theta \quad \cos \theta = \cos^2 \theta/2 - \sin^2 \theta/2$
 $= 1 - 2 \sin^2 \theta \quad = 1 - 2 \sin^2 \theta/2$
 $= 2 \cos^2 \theta - 1 \quad = 2 \cos^2 \theta/2 - 1$

$$= \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \quad = \frac{1 - \tan^2 \theta/2}{1 + \tan^2 \theta/2}$$

3. $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} \quad \tan \theta = \frac{2 \tan \theta/2}{1 - \tan^2 \theta/2}$

4. $\cot 2\theta = \frac{\cot^2 \theta - 1}{2 \cot \theta} \quad \cot \theta = \frac{\cot^2 \theta/2 - 1}{2 \cot \theta/2}$

$$5. \quad 1 + \cos 2\theta = 2\cos^2 \theta \quad 1 + \cos \theta = 2\cos^2 \theta/2$$

$$6. \quad 1 - \cos 2\theta = 2\sin^2 \theta \quad 1 - \cos \theta = 2 \sin^2 \theta/2$$

$$7. \quad \sin \theta = \pm \sqrt{\frac{1 - \cos 2\theta}{2}} \quad \sin \theta/2 = \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

$$8. \quad \cos \theta = \pm \sqrt{\frac{1 + \cos 2\theta}{2}} \quad \cos \theta/2 = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

$$9. \quad \tan \theta = \pm \sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}} \quad \tan \theta/2 = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$$

$$* \quad \begin{aligned} \sin 3\theta &= 3\sin \theta - 4\sin^3 \theta \\ \cos 3\theta &= 4\cos^3 \theta - 3\cos \theta \end{aligned}$$

$$\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

$$\cot 3\theta = \frac{3 \cot \theta - \cot^3 \theta}{1 - 3 \cot^2 \theta}$$

Transformations

$$* \quad \sin(A + B) + \sin(A - B) = 2\sin A \cos B$$

$$* \quad \sin(A + B) - \sin(A - B) = 2\cos A \sin B$$

$$* \quad \cos(A + B) + \cos(A - B) = 2\cos A \cos B$$

$$* \quad \cos(A - B) - \cos(A + B) = 2\sin A \sin B$$

II.

$$* \quad \sin C + \sin D = 2\sin\left(\frac{C + D}{2}\right) \cos\left(\frac{C - D}{2}\right)$$

$$* \quad \sin C - \sin D = 2\cos\left(\frac{C + D}{2}\right) \sin\left(\frac{C - D}{2}\right)$$

$$* \quad \cos C + \cos D = 2\cos\left(\frac{C + D}{2}\right) \cos\left(\frac{C - D}{2}\right)$$

$$* \quad \cos C - \cos D = -2\sin\left(\frac{C + D}{2}\right) \sin\left(\frac{C - D}{2}\right)$$

Some Important Problems with Solutions

1. Simplify the following problems

i. $\cos 315^\circ = \cos(360^\circ - 45^\circ) = \cos 45^\circ = \frac{1}{\sqrt{2}}$

ii. $\cot(-300^\circ) = -\cot 300^\circ = -\cot(360^\circ - 60^\circ) = -\cot(-60^\circ) = \frac{1}{\sqrt{3}}$

iii. $\sin\left(\frac{5\pi}{3}\right) = \sin\left(2\pi - \frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

2. Find the value of $\cos^2 45^\circ + \cos^2 135^\circ + \cos^2 225^\circ + \cos^2 315^\circ$.

Sol: $\begin{aligned} \cos^2 45^\circ + \cos^2 135^\circ + \cos^2 225^\circ + \cos^2 315^\circ \\ &= \cos^2 45^\circ + \cos^2(180^\circ - 45^\circ) + \cos^2(180^\circ + 45^\circ) + \cos^2(360^\circ - 45^\circ) \\ &= \cos^2 45^\circ + \cos^2 45^\circ + \cos^2 45^\circ + \cos^2 45^\circ \\ &= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2 \end{aligned}$

3. Find the value of $\cot \frac{\pi}{20} \cdot \cot \frac{3\pi}{20} \cdot \cot \frac{5\pi}{20} \cdot \cot \frac{7\pi}{20} \cdot \cot \frac{9\pi}{20}$.

Sol: $\begin{aligned} \cot \frac{\pi}{20} \cdot \cot \frac{3\pi}{20} \cdot \cot \frac{5\pi}{20} \cdot \cot \frac{7\pi}{20} \cdot \cot \frac{9\pi}{20} &= \cot 9^\circ \cdot \cot 27^\circ \cdot \cot 45^\circ \cdot \cot 63^\circ \cdot \cot 81^\circ \\ &= \cot 9^\circ \cdot \cot 27^\circ \cdot \cot(90^\circ - 27^\circ) \cdot \cot(90^\circ - 9^\circ) \\ &= \cot 9^\circ \cdot \cot 27^\circ \cdot \tan 27^\circ \cdot \tan 9^\circ = 1 \end{aligned}$

4. Find the value of $\sin 330^\circ \cdot \cos 120^\circ + \cos 210^\circ \cdot \sin 300^\circ$.

Sol: $\begin{aligned} \sin 330^\circ \cdot \cos 120^\circ + \cos 210^\circ \cdot \sin 300^\circ \\ &= \sin(360^\circ - 30^\circ) \cos(180^\circ - 60^\circ) + \cos(180^\circ + 30^\circ) \sin(360^\circ - 60^\circ) \\ &= (-\sin 30^\circ)(-\cos 60^\circ) + (-\cos 30^\circ)(-\sin 60^\circ) \\ &= \frac{1}{2} \cdot \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{1}{4} + \frac{3}{4} = 1 \end{aligned}$

5. If $\sin \alpha + \operatorname{cosec} \alpha = 2$, $n \in \mathbb{Z}$, then find the value of $\sin^n \alpha + \operatorname{cosec}^n \alpha$.

Sol: $\sin \alpha + \operatorname{cosec} \alpha = 2$

$$\sin \alpha + \frac{1}{\sin \alpha} = 2$$

$$\Rightarrow \sin^2 \alpha + 1 = 2 \sin \alpha$$

$$\Rightarrow (1 - \sin \alpha)^2 = 0$$

$$\Rightarrow 1 - \sin \alpha = 0$$

$$\Rightarrow \sin \alpha = 1 \Rightarrow \alpha = 90^\circ$$

$$\therefore \sin^n \alpha + \operatorname{Cosec}^n \alpha = \sin^n 90^\circ + \operatorname{Cosec}^n 90^\circ = 1^n + 1^n = 1 + 1 = 2$$

6. Eliminate θ from the following.

$$(i) x = a \cos^3 \theta ; y = b \sin^3 \theta$$

$$\text{Sol: } \frac{x}{a} = \cos^3 \theta \quad \frac{y}{b} = \sin^3 \theta$$

$$\cos \theta = \left(\frac{x}{a} \right)^{1/3} \quad \sin \theta = \left(\frac{y}{b} \right)^{1/3}$$

$$\therefore \cos^2 \theta + \sin^2 \theta = 1$$

$$\Rightarrow \left(\frac{x}{a} \right)^{2/3} + \left(\frac{y}{b} \right)^{2/3} = 1$$

$$ii. \quad x = a(\sec \theta + \tan \theta); y = b(\sec \theta - \tan \theta)$$

$$xy = ab(\sec^2 \theta - \tan^2 \theta)$$

$$= ab(1)$$

$$xy = ab$$

7. Find the period of the following functions.

$$i) \quad \cos(3x + 5) + 7$$

$$f(x) = \cos(3x + 5) + 7$$

$$\text{Period} = \frac{p}{|a|} = \frac{2\pi}{3}$$

$$ii) \quad \tan 5x$$

$$f(x) = \tan 5x$$

$$\text{Period} = \frac{\pi}{5}$$

$$iii) \quad \cos\left(\frac{4x+9}{5}\right)$$

$$f(x) = \cos\left(\frac{4x+9}{5}\right);$$

$$\text{Period} = \frac{2\pi}{4/5} = \frac{10\pi}{4} = \frac{5\pi}{2}$$

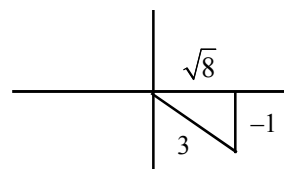
8. θ is not in 3rd quadrant, if $\sin\theta = -\frac{1}{3}$ then find the values of a) $\cos\theta$ b) $\cot\theta$.

Sol: $\sin\theta = -\frac{1}{3} < 0; \theta \notin Q_3.$

$$\Rightarrow \theta \in Q_4.$$

$$\text{a) } \cos\theta = \frac{\sqrt{8}}{3}$$

$$\text{b) } \cot\theta = -\sqrt{8}$$



9. Find the value of $\sin^2 82\frac{1}{2}^\circ - \sin^2 22\frac{1}{2}^\circ$.

Sol: $\sin^2 82\frac{1}{2}^\circ - \sin^2 22\frac{1}{2}^\circ = \sin^2 A - \sin^2 B$ Put $A = 82\frac{1}{2}^\circ$; $B = 22\frac{1}{2}^\circ$

$$= \sin(A+B) \sin(A-B)$$

$$= \sin 105^\circ \cdot \sin 60^\circ$$

$$= \sin(90 + 15^\circ) \sin 60^\circ$$

$$= \cos 15^\circ \cdot \sin 60^\circ$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{3+\sqrt{3}}{4\sqrt{2}}$$

10. Find the value of $\cos^2 112\frac{1}{2}^\circ - \sin^2 52\frac{1}{2}^\circ$

Sol: Let $A = 112\frac{1}{2}^\circ$; $B = 52\frac{1}{2}^\circ$

$$\cos^2 112\frac{1}{2}^\circ - \sin^2 52\frac{1}{2}^\circ = \cos^2 A - \sin^2 B$$

$$= \cos(A+B) \cdot \cos(A-B) \Rightarrow \cos(165^\circ) \cdot \cos 60^\circ$$

$$= \cos(180 - 15^\circ) \cdot \cos 60^\circ$$

$$= -\cos 15^\circ \cdot \cos 60^\circ$$

$$= -\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\left(\frac{1}{2}\right)$$

$$= -\left(\frac{\sqrt{3}+1}{4\sqrt{2}}\right)$$

11. Find the minimum and maximum value of the function $3\cos x + 4\sin x$.

Sol: Let $f(x) = 3\cos x + 4\sin x$

$$\text{Minimum value} = c - \sqrt{a^2 + b^2}$$

$$= 0 - \sqrt{4^2 + 3^2}$$

$$= -\sqrt{25}$$

$$= -5$$

$$\text{Maximum value} = c + \sqrt{a^2 + b^2}$$

$$= 0 + \sqrt{4^2 + 3^2}$$

$$= \sqrt{25}$$

$$= 5$$

12. Find the minimum and maximum value of the function $\sin 2x - \cos 2x$.

Sol: Let $f(x) = \sin 2x - \cos 2x$

$$\text{Minimum value} = c - \sqrt{a^2 + b^2}$$

$$= -\sqrt{1^2 + (-1)^2}$$

$$= -\sqrt{2}$$

$$\text{Maximum value} = c + \sqrt{a^2 + b^2}$$

$$= \sqrt{1^2 + (-1)^2}$$

$$= \sqrt{1+1}$$

$$= \sqrt{2}$$

13. Find the range of the function $7\cos x - 24\sin x + 5$.

Sol: Let $f(x) = 7\cos x - 24\sin x + 5$

$$\text{Minimum value of } f(x) = c - \sqrt{a^2 + b^2}$$

$$= 5 - \sqrt{(-24)^2 + 7^2}$$

$$= 5 - \sqrt{576 + 49}$$

$$= 5 - \sqrt{625}$$

$$= 5 - 25$$

$$= -20$$

$$\begin{aligned}
 \text{Maximum value of } f(x) &= c + \sqrt{a^2 + b^2} \\
 &= 5 + \sqrt{625} \\
 &= 5 + 25 \\
 &= 30
 \end{aligned}$$

14. If $\tan 20^\circ = P$, then prove that $\frac{\tan 610^\circ + \tan 700^\circ}{\tan 560^\circ - \tan 470^\circ} = \frac{1 - P^2}{1 + P^2}$.

Sol:
$$\begin{aligned}
 \frac{\tan 610^\circ + \tan 700^\circ}{\tan 560^\circ - \tan 470^\circ} &= \frac{\tan(360^\circ + 250^\circ) + \tan(360^\circ + 340^\circ)}{\tan(360^\circ + 200^\circ) - \tan(360^\circ + 110^\circ)} \\
 &= \frac{\tan 250^\circ + \tan 340^\circ}{\tan 200^\circ - \tan 110^\circ} \\
 &= \frac{\tan(270^\circ - 20^\circ) + \tan(360^\circ - 20^\circ)}{\tan(180^\circ + 20^\circ) - \tan(90^\circ + 20^\circ)} \\
 &= \frac{\cot 20^\circ - \tan 20^\circ}{\tan 20^\circ + \cot 20^\circ} \\
 &= \frac{\frac{1}{p} - p}{p + \frac{1}{p}} = \frac{1 - p^2}{1 + p^2} = \text{RHS}
 \end{aligned}$$

15. Prove that $\frac{\tan \theta + \sec \theta - 1}{\tan \theta - \sec \theta + 1} = \frac{1 + \sin \theta}{\cos \theta}$.

Sol:
$$\begin{aligned}
 \text{LHS} &= \frac{\tan \theta + \sec \theta - 1}{\tan \theta - \sec \theta + 1} \\
 &= \frac{\tan \theta + \sec \theta - (\sec^2 \theta - \tan^2 \theta)}{\tan \theta - \sec \theta + 1} \\
 &= \frac{(\tan \theta + \sec \theta)(1 - \sec \theta + \tan \theta)}{(\tan \theta - \sec \theta + 1)} \\
 &= \tan \theta + \sec \theta \\
 &= \frac{\sin \theta}{\cos \theta} + \frac{1}{\cos \theta} \\
 &= \frac{1 + \sin \theta}{\cos \theta} = \text{RHS}
 \end{aligned}$$

16. Prove that $(1 + \cot\theta - \operatorname{cosec}\theta)(1 + \tan\theta + \sec\theta) = 2$.

Sol: LHS = $(1 + \cot\theta - \operatorname{cosec}\theta)(1 + \tan\theta + \sec\theta)$

$$= \left(1 + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right) \left(1 + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)$$

$$= \left(\frac{\sin\theta + \cos\theta - 1}{\sin\theta}\right) \left(\frac{\cos\theta + \sin\theta + 1}{\cos\theta}\right)$$

$$= \frac{(\sin\theta + \cos\theta)^2 - 1}{\sin\theta \cos\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1}{\sin\theta \cos\theta}$$

$$= \frac{1 + 2\sin\theta\cos\theta - 1}{\sin\theta \cos\theta}$$

$$= \frac{2\sin\theta\cos\theta}{\sin\theta \cos\theta}$$

$$= 2 = \text{RHS}$$

17. If θ is in 3rd Quadrant and $\tan\theta = \frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ}$, then find the value of θ .

Sol: $\tan\theta = \frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ}$

$$= \frac{\cos 11^\circ \left(1 + \frac{\sin 11^\circ}{\cos 11^\circ}\right)}{\cos 11^\circ \left(1 - \frac{\sin 11^\circ}{\cos 11^\circ}\right)}$$

$$= \frac{1 + \tan 11^\circ}{1 - \tan 11^\circ}$$

$$= \tan(45^\circ + 11^\circ)$$

$$= \tan 56^\circ$$

$$= \tan(180^\circ + 56^\circ) = \tan 236^\circ$$

$$= \theta = 236^\circ$$

18. Prove that $\frac{\cos 9^\circ + \sin 9^\circ}{\cos 9^\circ - \sin 9^\circ} = \cot 36^\circ$

Sol: LHS = $\frac{\cos 9^\circ + \sin 9^\circ}{\cos 9^\circ - \sin 9^\circ}$

$$\begin{aligned}
&= \frac{1 + \tan 9^\circ}{1 - \tan 9^\circ} \\
&= \tan(45^\circ + 9^\circ) \\
&= \tan 54^\circ \\
&= \tan(90^\circ - 36^\circ) \\
&= \cot 36^\circ = \text{RHS}
\end{aligned}$$

19. If $A + B = \frac{\pi}{4}$, then prove that $(1 + \tan A)(1 + \tan B) = 2$.

Sol: $A + B = 45^\circ$

$$= \tan(A + B) = \tan 45^\circ = 1$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$$

$$= \tan A + \tan B = 1 - \tan A \tan B$$

$$= \tan A + \tan B + \tan A \tan B = 1 \quad \dots\dots\dots(1)$$

$$\Rightarrow \text{Now } (1 + \tan A)(1 + \tan B) = 1 + \tan A + \tan B + \tan A \tan B = 2 \quad (\text{from 1})$$

20. Show that $\cos^2 \theta + \cos^2 \left(\frac{2\pi}{3} + \theta \right) + \cos^2 \left(\frac{2\pi}{3} - \theta \right) = \frac{3}{2}$.

Sol:

$$\begin{aligned}
&\cos^2 \left(\frac{2\pi}{3} + \theta \right) + \cos^2 \left(\frac{2\pi}{3} - \theta \right) \\
&= \cos^2(120^\circ + \theta) + \cos^2(120^\circ - \theta) \\
&= \cos^2(60^\circ + \theta) + \cos^2(60^\circ - \theta) \\
&= (\cos 60^\circ \cos \theta - \sin 60^\circ \sin \theta)^2 + (\cos 60^\circ \cos \theta + \sin 60^\circ \sin \theta)^2 \\
&= 2[\cos^2 60^\circ \cos^2 \theta + \sin^2 60^\circ \sin^2 \theta] \\
&[\because (a + b)^2 + (a - b)^2 = 2(a^2 + b^2)]
\end{aligned}$$

$$= 2 \left[\left(\frac{1}{2} \right)^2 \cos^2 \theta + \left(\frac{\sqrt{3}}{2} \right)^2 \sin^2 \theta \right]$$

$$= 2 \left[\frac{1}{4} \cos^2 \theta + \frac{3}{4} \sin^2 \theta \right]$$

$$= \frac{2}{4} [\cos^2 \theta + 3 \sin^2 \theta]$$

$$= \frac{1}{2} [\cos^2 \theta + 3 \sin^2 \theta]$$

$$= \text{LHS} = \cos^2\theta + \frac{1}{2}\cos^2\theta + \frac{3}{2}\sin^2\theta$$

$$= \frac{3}{2}\cos^2\theta + \frac{3}{2}\sin^2\theta$$

$$= \frac{3}{2}(\cos^2\theta + \sin^2\theta) = \frac{3}{2} = \text{RHS}$$

21. If $\frac{\sin\alpha}{a} = \frac{\cos\alpha}{b}$, then show that $a\sin 2\alpha + b\cos 2\alpha = b$.

Sol: $\frac{\sin\alpha}{a} = \frac{\cos\alpha}{b} = k$

$$\sin\alpha = ak, \cos\alpha = bk$$

$$\begin{aligned} \text{LHS} &= a\sin 2\alpha + b\cos 2\alpha \\ &= a(2\sin\alpha\cos\alpha) + b(1 - 2\sin^2\alpha) \\ &= a[2(ak)(bk)] + b[1 - 2(ak)^2] \\ &= 2a^2bk^2 + b - 2a^2bk^2 \\ &= b = \text{RHS} \end{aligned}$$

22. Prove that $\frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} = 4$.

Sol:
$$\begin{aligned} \text{LHS} &= \frac{1}{\sin 10^\circ} - \frac{\sqrt{3}}{\cos 10^\circ} \\ &= \frac{\cos 10^\circ - \sqrt{3}\sin 10^\circ}{\sin 10^\circ \cos 10^\circ} \\ &= \frac{2\left[\frac{1}{2}\cos 10^\circ - \frac{\sqrt{3}}{2}\sin 10^\circ\right]}{\frac{1}{2}(2\sin 10^\circ \cos 10^\circ)} \\ &= \frac{4[\sin 30^\circ \cos 10^\circ - \cos 30^\circ \sin 10^\circ]}{\sin 20^\circ} \\ &= \frac{4\sin(30^\circ - 10^\circ)}{\sin 20^\circ} \\ &= \frac{4\sin 20^\circ}{\sin 20^\circ} \\ &= 4 = \text{RHS} \end{aligned}$$

23. In a $\triangle ABC$, $\tan \frac{A}{2} = \frac{5}{6}$, $\tan \frac{B}{2} = \frac{20}{37}$, then show that $\tan \frac{C}{2} = \frac{2}{5}$.

Sol: $A + B + C = 180^\circ$

$$\begin{aligned}
 \tan\left(\frac{A+B}{2}\right) &= \tan\left(90 - \frac{C}{2}\right) \\
 &= \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \cot \frac{C}{2} \\
 &= \frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{1 - \tan \frac{A}{2} \tan \frac{B}{2}} = \cot \frac{C}{2} \\
 &= \frac{\frac{5}{6} + \frac{20}{37}}{1 - \frac{5}{6} \cdot \frac{20}{37}} = \cot \frac{C}{2} \\
 &\Rightarrow \frac{\frac{185+120}{222}}{\frac{222-100}{222}} = \cot \frac{C}{2} \\
 &\Rightarrow \frac{305}{122} = \frac{1}{\tan \frac{C}{2}} \\
 &\Rightarrow \tan \frac{C}{2} = \frac{122}{305} = \frac{2}{5} \\
 &\Rightarrow \tan \frac{C}{2} = \frac{2}{5}
 \end{aligned}$$

24. Prove that $\cos^2 \frac{\pi}{8} + \cos^2 \frac{3\pi}{8} + \cos^2 \frac{5\pi}{8} + \cos^2 \frac{7\pi}{8} = 2$.

$$\begin{aligned}
 \text{Sol: LHS} &= \cos^2 \frac{\pi}{8} + \cos^2 \frac{3\pi}{8} + \cos^2 \frac{5\pi}{8} + \cos^2 \frac{7\pi}{8} \\
 &= \cos^2 \frac{\pi}{8} + \cos^2 \frac{3\pi}{8} + \cos^2 \left(\pi - \frac{3\pi}{8}\right) + \cos^2 \left(\pi - \frac{\pi}{8}\right) \\
 &= \cos^2 \frac{\pi}{8} + \cos^2 \frac{3\pi}{8} + \cos^2 \frac{3\pi}{8} + \cos^2 \frac{\pi}{8}
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \left(\cos^2 \frac{\pi}{8} + \cos^2 \frac{3\pi}{8} \right) \\
 &= 2 \left(\cos^2 \frac{\pi}{8} + \cos^2 \left(\frac{\pi}{2} - \frac{\pi}{8} \right) \right) \\
 &= 2 \left(\cos^2 \frac{\pi}{8} + \sin^2 \frac{\pi}{8} \right) = 2(1) = 2 = \text{RHS}
 \end{aligned}$$

25. Show that $\sin \frac{\pi}{5} \cdot \sin \frac{2\pi}{5} \cdot \sin \frac{3\pi}{5} \cdot \sin \frac{4\pi}{5} = \frac{5}{16}$.

Sol: LHS = $\sin \frac{\pi}{5} \cdot \sin \frac{2\pi}{5} \cdot \sin \frac{3\pi}{5} \cdot \sin \frac{4\pi}{5}$

$$\begin{aligned}
 &= \sin 36^\circ \cdot \sin 72^\circ \cdot \sin 108^\circ \cdot \sin 144^\circ \\
 &= \sin 36^\circ \cdot \sin(90-18^\circ) \cdot \sin(90+18^\circ) \cdot \sin(180-36^\circ) \\
 &= \sin 36^\circ \cdot \cos 18^\circ \cdot \cos 18^\circ \cdot \sin 36^\circ \\
 &= \sin^2 36^\circ \cdot \cos^2 18^\circ \\
 &= \frac{10-2\sqrt{5}}{16} \cdot \frac{10+2\sqrt{5}}{16} \\
 &= \frac{100-20}{16 \times 16} = \frac{80}{16 \times 16} = \frac{5}{16} = \text{RHS}
 \end{aligned}$$

26. Prove that $\left(1 + \cos \frac{\pi}{10}\right) \left(1 + \cos \frac{3\pi}{10}\right) \left(1 + \cos \frac{7\pi}{10}\right) \left(1 + \cos \frac{9\pi}{10}\right) = \frac{1}{16}$.

Sol: LHS = $\left(1 + \cos \frac{\pi}{10}\right) \left(1 + \cos \frac{3\pi}{10}\right) \left(1 + \cos \frac{7\pi}{10}\right) \left(1 + \cos \frac{9\pi}{10}\right)$

$$\begin{aligned}
 &= \left(1 + \cos \frac{\pi}{10}\right) \left(1 + \cos \frac{3\pi}{10}\right) \left(1 + \cos \left(\pi - \frac{3\pi}{10}\right)\right) \left(1 + \cos \left(\pi - \frac{\pi}{10}\right)\right) \\
 &= \left(1 + \cos \frac{\pi}{10}\right) \left(1 + \cos \frac{3\pi}{10}\right) \left(1 - \cos \frac{3\pi}{10}\right) \left(1 - \cos \frac{\pi}{10}\right) \\
 &= \left(1 - \cos^2 \frac{\pi}{10}\right) \left(1 - \cos^2 \frac{3\pi}{10}\right) \\
 &= \sin^2 \frac{\pi}{10} \sin^2 \frac{3\pi}{10} \\
 &= \sin^2 18^\circ \cdot \sin^2 54^\circ \\
 &= \left(\frac{\sqrt{5}-1}{4}\right)^2 \left(\frac{\sqrt{5}+1}{4}\right)^2
 \end{aligned}$$

$$= \left(\frac{(\sqrt{5}-1)(\sqrt{5}+1)}{16 \times 16} \right)^2$$

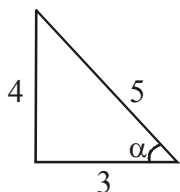
$$= \frac{(5-1)^2}{16 \times 16} = \frac{16}{16 \times 16} = \frac{1}{16} = \text{RHS}$$

27. If α, β angles, $\text{Cos}\alpha = \frac{3}{5}$, $\text{Cos}\beta = \frac{5}{13}$, then show that

(i) $\text{Sin}^2\left(\frac{\alpha - \beta}{2}\right) = \frac{1}{65}$, (ii) $\text{Cos}^2\left(\frac{\alpha + \beta}{2}\right) = \frac{16}{65}$

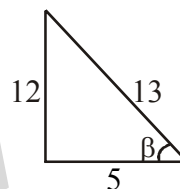
Sol: $\text{Cos}\alpha = \frac{3}{5}$

$\text{Sin}\alpha = \frac{4}{5}$



$\text{Cos}\beta = \frac{5}{13}$

$\text{Sin}\beta = \frac{12}{13}$



(i) $2\text{Sin}^2\left(\frac{\alpha - \beta}{2}\right) = 1 - \text{Cos}(\alpha - \beta) \quad \therefore 2\text{Sin}^2\theta = 1 - \text{Cos}2\theta$

$$= 1 - [\text{Cos}\alpha \text{Cos}\beta + \text{Sin}\alpha \text{Sin}\beta]$$

$$= 1 - \left[\frac{3}{5} \cdot \frac{5}{13} + \frac{4}{5} \cdot \frac{12}{13} \right]$$

$$= 1 - \frac{15}{65} - \frac{48}{65}$$

$$= \frac{65 - 15 - 48}{65}$$

$$= \frac{65 - 63}{65} = \frac{2}{65}$$

$$\therefore \text{Sin}^2\left(\frac{\alpha - \beta}{2}\right) = \frac{1}{65}$$

(ii) $2\text{Cos}^2\left(\frac{\alpha + \beta}{2}\right) = 1 + \text{Cos}(\alpha + \beta) \quad \therefore 2\text{Cos}^2\theta = 1 + \text{Cos}2\theta$

$$= 1 + \text{Cos}\alpha \text{Cos}\beta - \text{Sin}\alpha \text{Sin}\beta$$

$$= 1 + \left[\frac{3}{5} \cdot \frac{5}{13} + \frac{4}{5} \cdot \frac{12}{13} \right]$$

$$= 1 + \frac{15}{65} - \frac{48}{65}$$

$$= \frac{65+15-48}{65}$$

$$= \frac{80-48}{65}$$

$$\therefore 2\cos^2\left(\frac{\alpha - \beta}{2}\right) = \frac{32}{65}$$

$$\cos^2\left(\frac{\alpha + \beta}{2}\right) = \frac{16}{65}$$

28. A, B, C are angles in a triangle. Then prove that

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

Sol: $\text{LHS} = \sin 2A + \sin 2B + \sin 2C$
 $= 2\sin(A+B)\cos(A-B) + \sin 2C$
 $= 2\sin C \cos(A-B) + 2\sin C \cos C$
 $= 2\sin C [\cos(A-B) + \cos C]$
 $= 2\sin C [\cos(A-B) - \cos(A+B)]$
 $= 2\sin C [2\sin A \sin B]$
 $= 4\sin A \sin B \sin C$
 $= \text{RHS}$

29. Prove that $\cos 2A - \cos 2B + \cos 2C = 1 - 4\sin A \cos B \sin C$.

Sol: $\cos 2A - \cos 2B + \cos 2C$
 $= -2\sin(A+B)\sin(A-B) + \cos 2C$
 $= -2\sin C \sin(A-B) + 1 - 2\sin^2 C$
 $= 1 - 2\sin C [\sin(A-B) + \sin C]$
 $= 1 - 2\sin C [\sin(A-B) + \sin(A+B)]$
 $= 1 - 2\sin C [2\sin A \cos B]$
 $= 1 - 4\sin A \cos B \sin C$

30. A, B, C are angles in a triangle. Then prove that

$$\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \cdot \sin \frac{B}{2} \cdot \cos \frac{C}{2}$$

Sol: $\text{LHS} = \sin A + \sin B - \sin C$
 $= 2\sin \frac{(A+B)}{2} \cos \frac{A-B}{2} - \sin C$

$$\begin{aligned}
&= 2\cos\frac{C}{2}\cos\frac{A-B}{2} - 2\sin\frac{C}{2}\cos\frac{C}{2} \\
&= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\frac{C}{2}\right] \\
&= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right] \\
&= 2\cos\frac{C}{2}\left[\sin\frac{A}{2} \cdot \sin\frac{B}{2}\right] \\
&= 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} \\
&= \text{RHS}
\end{aligned}$$

$$\begin{aligned}
&\left[\because \sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{180-C}{2}\right) \right. \\
&\quad \left. = \sin\left(90 - \frac{C}{2}\right) = \cos\frac{C}{2} \right] \\
&\left(\because \sin\frac{C}{2} = \sin\left(90 - \frac{A+B}{2}\right) \right) \\
&\quad = \cos\left(\frac{A+B}{2}\right)
\end{aligned}$$

31. Prove that $\cos A + \cos B - \cos C = -1 + 4 \cos\frac{A}{2} \cos\frac{B}{2} \sin\frac{C}{2}$

Sol: LHS = $\cos A + \cos B - \cos C$

$$\begin{aligned}
&= 2\cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) - \cos C \\
&= 2\sin\frac{C}{2}\cos\frac{A-B}{2} - \left(1 - 2\sin^2\frac{C}{2}\right) \\
&= 2\sin\frac{C}{2}\cos\frac{A-B}{2} - 1 + 2\sin^2\frac{C}{2} \\
&= -1 + 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\right] \\
&= -1 + 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right] \\
&= -1 + 2\sin\frac{C}{2}\left[2\cos\frac{A}{2} \cdot \cos\frac{B}{2}\right] \\
&= -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} = \text{RHS}
\end{aligned}$$

- 32.** A, B, C are angles in a triangle. Then prove that
 $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C$.

Sol: $\text{LHS} = \sin^2 A + \sin^2 B - \sin^2 C$

$$= 1 - \cos^2 A + \sin^2 B - \sin^2 C$$

$$= 1 - (\cos^2 A - \sin^2 B) - \sin^2 C$$

$$= 1 - \cos(A + B) \cos(A - B) - 1 + \cos^2 C$$

$$= \cos C \cos(A - B) + \cos^2 C$$

$$= \cos C [\cos C + \cos(A - B)]$$

$$= +\cos C [\cos(A - B) - \cos(A + B)]$$

$$= \cos C [2 \sin A \sin B]$$

$$= 2 \sin A \sin B \cos C$$

$$= \text{RHS}$$

- 33.** A, B, C are angles in a triangle. Then prove that
 $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C$

Sol: $\text{LHS} = \cos^2 A + \cos^2 B - \cos^2 C$

$$= \cos^2 A + 1 - \sin^2 B - \cos^2 C$$

$$= 1 + (\cos^2 A - \sin^2 B) - \cos^2 C$$

$$= 1 + \cos(A + B) \cos(A - B) - \cos^2 C$$

$$= 1 - \cos C \cos(A - B) - \cos^2 C$$

$$= 1 - \cos C [\cos(A - B) + \cos C]$$

$$= 1 - \cos C [\cos(A - B) - \cos(A + B)]$$

$$= 1 - \cos C [2 \sin A \sin B]$$

$$= 1 - 2 \sin A \sin B \cos C$$

$$= \text{RHS}$$



Hyperbolic Equations

1. $\forall x \in \mathbb{R} \quad \sinh x = \frac{e^x - e^{-x}}{2}$
2. $\forall x \in \mathbb{R} \quad \cosh x = \frac{e^x + e^{-x}}{2}$
3. $\forall x \in \mathbb{R} \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
4. $\forall x \in \mathbb{R} - \{0\} \quad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
5. $\forall x \in \mathbb{R} \quad \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$
6. $\forall x \in \mathbb{R} - \{0\} \quad \operatorname{cosech} x = \frac{2}{e^x - e^{-x}}$

Note:

- 1) $\cosh(0) = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = \frac{2}{2} = 1$
- 2) $\sinh(0) = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = \frac{0}{2} = 0$
- 3) $\sinh(-x) = \frac{e^x + e^{-(-x)}}{2} = \frac{e^{-x} + e^x}{2} = \cosh x$

$$f(-x) = f(x)$$

$\therefore \cosh x$ is an even function.

$$\begin{aligned}
 (4) \quad \sinh(-x) &= \frac{e^x - e^{-(-x)}}{2} = \frac{e^{-x} - e^x}{2} \\
 &= -\left(\frac{e^x - e^{-x}}{2}\right) = -\sinh x
 \end{aligned}$$

$\Rightarrow f(-x) = -f(x) \quad \therefore f(x) = \sinh x$ is an odd function

IDENTITIES

1. $\forall x \in \mathbb{R} \quad \cosh^2 x - \sinh^2 x = 1$
2. $\forall x \in \mathbb{R} \quad 1 - \tanh^2 x = \operatorname{sech}^2 x$
3. $\forall x \in \mathbb{R} \quad \coth^2 x - 1 = \operatorname{cosech}^2 x$

Theorem - 1

- (i) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
- (ii) $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$
- (iii) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
- (iii) $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

2. $\forall x \in \mathbb{R}$

$$(i) \quad \sinh 2x = 2 \sinh x \cosh x = \frac{2 \tanh x}{1 - \tanh^2 x}$$

$$(ii) \quad \begin{aligned} \cosh 2x &= 2 \cosh^2 x - 1 \\ &= 1 + 2 \sinh^2 x \\ &= \frac{1 + \tanh^2 x}{1 - \tanh^2 x} \\ &= \cosh^2 x + \sinh^2 x \end{aligned}$$

3. $\forall x, y \in \mathbb{R}$

$$(i) \quad \tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

$$(ii) \quad \tanh(x - y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$$

4. $\forall x \in \mathbb{R}$

$$(i) \quad \tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

$$(ii) \quad \coth 2x = \frac{\coth^2 x + 1}{2 \coth x}$$

Theorem: $\forall x \in \mathbb{R}$

$$\sinh^{-1} x = \log_e (x + \sqrt{x^2 + 1})$$

Theorem: $\forall x \in [1, \alpha]$

$$\cosh^{-1} x = \log_e (x + \sqrt{x^2 - 1})$$

Theorem: $\forall x \in [-1, 1]$

$$\tanh^{-1} x = \frac{1}{2} \log_e \left(\frac{1+x}{1-x} \right)$$

PROBLEMS

1. If $\sinh x = \frac{3}{4}$ Then find $\cosh(2x)$, $\sinh(2x)$

Sol: $\cosh^2 x = 1 + \sinh^2 x$

$$= 1 + \left(\frac{3}{4}\right)^2$$

$$= 1 + \frac{9}{16}$$

$$= \frac{25}{16}$$

$$\cosh x = \frac{5}{4}$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$= \left(\frac{5}{4}\right)^2 + \left(\frac{3}{4}\right)^2$$

$$= \frac{25}{16} + \frac{9}{16}$$

$$= \frac{34}{16} = \frac{17}{8}$$

$$\sinh 2x = 2 \sinh x \cosh x = 2 \left(\frac{3}{4}\right) \left(\frac{5}{4}\right) = \frac{15}{8}$$

2. If $\sinh x = 3$ Then show that $x = \log_e(3 + \sqrt{10})$

Sol: $\sinh x = 3$

$$x = \sinh^{-1}(3)$$

$$= \log_e(3 + \sqrt{3^2 + 1}) \quad \because \sinh^{-1} x = \log_e(x + \sqrt{x^2 + 1})$$

$$x = \log_e(3 + \sqrt{10})$$

3. $\forall n \in \mathbb{R}$

$$(i) (\cosh x - \sinh x)^n = \cosh(nx) - \sinh(nx)$$

$$(ii) (\cosh x + \sinh x)^n = \cosh(nx) + \sinh(nx)$$

Sol: $(\cosh x - \sinh x)^n = \left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} \right)^n$

$$= \left(\frac{e^x + e^{-x} - e^x + e^{-x}}{2} \right)^n$$

$$\begin{aligned}
 &= \left(\frac{2e^{-x}}{2} \right)^n \\
 &= e^{-nx} \\
 &= \left(\frac{e^{nx} + e^{-nx}}{2} \right) - \left(\frac{e^{nx} - e^{-nx}}{2} \right) \\
 &= \text{Cosh}(nx) - \text{Sinh}(nx)
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad (\text{Cosh}x + \text{Sinh}x)^n &= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} \right)^n \\
 &= \left(\frac{e^x + e^{-x} + e^x - e^{-x}}{2} \right)^n \\
 &= \left(\frac{2e^x}{2} \right)^n \\
 &= e^{nx} \\
 &= \left(\frac{e^{nx} + e^{-nx}}{2} \right) + \left(\frac{e^{nx} - e^{-nx}}{2} \right) \\
 &= \text{Cosh}(nx) + \text{Sinh}(nx)
 \end{aligned}$$

4. If $\forall x \in \mathbb{R}$ Then has that **Cosh⁴x – Sinh⁴x = Cosh(2x)**

$$\begin{aligned}
 \text{Sol: } \text{Cosh}^4x - \text{Sinh}^4x &= (\text{Cosh}^2x + \text{Sinh}^2x)(\text{Cosh}^2x - \text{Sinh}^2x) \\
 &= \text{Cosh}(2x) \quad (1) \\
 &= \text{Cosh}(2x)
 \end{aligned}$$

5. If Than has that $\text{Tanh}^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\log_e 3$

$$\text{Sol: } \text{Tanh}^{-1}x = \frac{1}{2}\log_e \left(\frac{1+x}{1-x} \right)$$

$$\begin{aligned}
 \text{Tanh}^{-1}\left(\frac{1}{2}\right) &= \frac{1}{2}\log_e \left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}} \right) \\
 &= \frac{1}{2}\log_e \left(\frac{3/2}{1/2} \right) \\
 &= \frac{1}{2}\log_e 3
 \end{aligned}$$



PROPERTIES OF TRIANGLES

Important Points - Formulas

1. In $\triangle ABC$, $BC = a$, $CA = b$, $AB = c$

$$a + b + c = 2S \Rightarrow S = \frac{a + b + c}{2}$$

2. **Sine Rule:** In $\triangle ABC$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

R – circumcentre of $\triangle ABC$.

3. **Cosine Rule:** $a^2 = b^2 + c^2 - 2bc \cos A$

$$b^2 = c^2 + a^2 - 2ca \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}; \cos B = \frac{c^2 + a^2 - b^2}{2ca}; \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

4. **Projection Rule:**

$$a = b \cos C + c \cos B$$

$$b = c \cos A + a \cos C$$

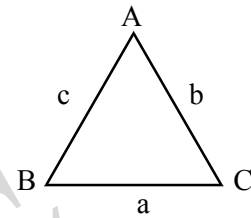
$$c = a \cos B + b \cos A$$

5. **Tangent or Napier's Rule:**

$$\tan\left(\frac{B - C}{2}\right) = \left(\frac{b - c}{b + c}\right) \cot \frac{A}{2}$$

$$\tan\left(\frac{A - B}{2}\right) = \left(\frac{a - b}{a + b}\right) \cot \frac{C}{2}$$

$$\tan\left(\frac{C - A}{2}\right) = \left(\frac{c - a}{c + a}\right) \cot \frac{B}{2}$$



$$6. \quad \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}; \sin \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}; \sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}; \cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}; \cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}; \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}; \tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$7. \quad \Delta ABC \text{ Area} \rightarrow \Delta = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \frac{1}{2} ab \sin C$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = 2R^2 \sin A \sin B \sin C.$$

$$8. \quad \tan \frac{A}{2} = \frac{(s-b)(s-c)}{\Delta}; \tan \frac{B}{2} = \frac{(s-a)(s-c)}{\Delta}; \tan \frac{C}{2} = \frac{(s-a)(s-b)}{\Delta}$$

$$\cot \frac{A}{2} = \frac{s(s-a)}{\Delta}; \cot \frac{B}{2} = \frac{s(s-b)}{\Delta}; \cot \frac{C}{2} = \frac{s(s-c)}{\Delta}$$

$$9. \quad r = \frac{\Delta}{s}; r_1 = \frac{\Delta}{s-a}; r_2 = \frac{\Delta}{s-b}; r_3 = \frac{\Delta}{s-c}$$

r - radius of incircle

r_1, r_2, r_3 - radii of excircles.

$$10. \quad r = \frac{\Delta}{s} = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$11. \quad r_1 = \frac{\Delta}{s-a} = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = S \tan \frac{A}{2}$$

$$12. \quad r_2 = \frac{\Delta}{s-b} = 4R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} = S \tan \frac{B}{2}$$

$$13. \quad r_3 = \frac{\Delta}{s-c} = 4R \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2} = S \tan \frac{C}{2}$$

$$14. \quad \Delta^2 = r r_1 r_2 r_3$$

$$15. \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

Short & Long Answer Questions

(Note: In all problems are refer to ΔABC)

$$1. \quad \text{In } \Delta ABC, \text{ if } a = 3, b = 4 \text{ and } \sin A = \frac{3}{4} \text{ then find angle } B.$$

$$\text{Sol: From Sine Rule } \frac{a}{\sin A} = \frac{b}{\sin B}$$

$$a \sin B = b \sin A$$

$$\sin B = \frac{b \sin A}{a} = \frac{4 \left(\frac{3}{4} \right)}{3} = 1 \quad \left(\because \text{from assumption } b = 4; a = 3; \sin A = \frac{3}{4} \right)$$

$$\sin B = 1 = \sin 90^\circ$$

$$\angle B = 90^\circ$$

2. If $a = 26$ cm; $b = 30$ cm and $\cos C = \frac{63}{65}$ then find the value of c .

Sol: From Cosine rule $c^2 = a^2 + b^2 - 2ab \cos C$

$$c^2 = (26)^2 + (30)^2 - 2(26)(30) \left(\frac{63}{65} \right) \quad \left(\because \text{from assumption rule } a = 26 \text{ cm; } b = 30 \text{ cm,} \right.$$

$$\left. \cos C = \frac{63}{65} \right)$$

$$= 676 + 900 - 1512 = 64$$

$$c^2 = 64$$

$$c = 8$$

3. Show that $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$.

Sol: LHS = $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C$
 $= b \cos A + c \cos A + c \cos B + a \cos B + a \cos C + b \cos C$
 $= (a \cos B + b \cos A) + (b \cos C + c \cos B) + (c \cos A + a \cos C)$
 $= c + a + b \quad \left(\because \text{from projection rule} \right)$
 $= a + b + c = \text{RHS}$

$$\therefore (b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$$

4. Show that $b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2} = S$

Sol: LHS = $b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2}$
 $= b \left[\frac{s(s - c)}{ab} \right] + c \left[\frac{s(s - b)}{ac} \right]$
 $= \frac{s(s - c)}{a} + \frac{s(s - b)}{a} = \frac{s}{a} [s - c + s - b] = \frac{s}{a} [2s - b - c]$
 $= \frac{s}{a} [a + b + c - b - c]$
 $= \frac{s}{a} [a] = S = \text{RHS}$

$$\therefore b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2} = S$$

5. Show that $\frac{a}{bc} + \frac{\cos A}{a} = \frac{b}{ca} + \frac{\cos B}{b} = \frac{c}{ab} + \frac{\cos C}{c}$.

Sol: $\frac{a}{bc} + \frac{\cos A}{a} = \frac{a}{bc} + \frac{\left(\frac{b^2 + c^2 - a^2}{2bc}\right)}{a} \quad (\because \text{from Cosine rule } \cos A = \frac{b^2 + c^2 - a^2}{2bc})$

$$= \frac{a}{bc} + \frac{b^2 + c^2 - a^2}{2abc} = \frac{2a^2 + b^2 + c^2 - a^2}{2abc} = \frac{a^2 + b^2 + c^2}{2abc}$$

Similarly, $\frac{b}{ca} + \frac{\cos B}{b} = \frac{a^2 + b^2 + c^2}{2abc}$

$$\frac{c}{ab} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

$$\therefore \frac{a}{bc} + \frac{\cos A}{a} = \frac{b}{ca} + \frac{\cos B}{b} = \frac{c}{ab} + \frac{\cos C}{c}$$

6. Show that $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$.

Sol: LHS = $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$

$$= \frac{\left(\frac{b^2 + c^2 - a^2}{2bc}\right)}{a} + \frac{\left(\frac{c^2 + a^2 - b^2}{2ca}\right)}{b} + \frac{\left(\frac{a^2 + b^2 - c^2}{2ab}\right)}{c} \quad (\because \text{from Cosine rule})$$

$$= \frac{b^2 + c^2 - a^2}{2abc} + \frac{c^2 + a^2 - b^2}{2abc} + \frac{a^2 + b^2 - c^2}{2abc}$$

$$= \frac{b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2}{2abc} = \frac{a^2 + b^2 + c^2}{2abc} = \text{RHS}$$

$$\therefore \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

7. Write the value of $a \sin^2 \frac{C}{2} + c \sin^2 \frac{A}{2}$ in terms of s, a, b, c.

Sol: $a \sin^2 \frac{C}{2} + c \sin^2 \frac{A}{2} = a \left[\frac{(s-a)(s-b)}{ab} \right] + c \left[\frac{(s-b)(s-c)}{bc} \right]$

$$= \frac{(s-a)(s-b)}{b} + \frac{(s-b)(s-c)}{b}$$

$$= \frac{s-b}{b} [s-a+s-c] = \frac{s-b}{b} [2s-a-c] = \frac{s-b}{b} [a+b+c-a-c]$$

$$= \frac{s-b}{b} [b] = s-b$$

$$\therefore a \sin^2 \frac{C}{2} + c \sin^2 \frac{A}{2} = s-b$$

8. Prove that $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \frac{s^2}{\Delta}$.

Sol: LHS = $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}$

$$= \frac{s(s-a)}{\Delta} + \frac{s(s-b)}{\Delta} + \frac{s(s-c)}{\Delta} = \frac{s}{\Delta} [s-a+s-b+s-c]$$

$$= \frac{s}{\Delta} [3s-(a+b+c)] = \frac{s}{\Delta} [3s-2s]$$

$$= \frac{s}{\Delta} \cdot [s] = \frac{s^2}{\Delta} = \text{RHS}$$

$$\therefore \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \frac{s^2}{\Delta}$$

9. Prove that, $\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} = \frac{bc+ca+ab-s^2}{\Delta}$.

Sol: LHS = $\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} = \frac{(s-b)(s-c)}{\Delta} + \frac{(s-a)(s-c)}{\Delta} + \frac{(s-a)(s-b)}{\Delta}$

$$= \frac{s^2 - s(b+c) + bc + s^2 - s(a+c) + ac + s^2 - s(a+b) + ab}{\Delta}$$

$$= \frac{3s^2 - 2s(a+b+c) + ab+bc+ca}{\Delta} = \frac{ab+bc+ca-s^2}{\Delta} = \text{RHS}$$

$$\therefore \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} = \frac{bc+ca+ab-s^2}{\Delta}$$

10. If $\sin \theta = \frac{a}{b+c}$, show that $\cos \theta = \frac{2\sqrt{bc}}{b+c} - \cos \frac{A}{2}$.

Sol: Given $\sin \theta = \frac{a}{b+c}$ (1)

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{a}{b+c}\right)^2 \quad [\because \text{from (1)}]$$

$$= 1 - \frac{a^2}{(b+c)^2} = \frac{(b+c)^2 - a^2}{(b+c)^2} = \frac{(b+c+a)(b+c-a)}{(b+c)^2}$$

$$= \frac{2s(2s-a-a)}{(b+c)^2} = \frac{2s \cdot 2(s-a)}{bc} \cdot \frac{bc}{(b+c)^2}$$

$$\cos^2\theta = \frac{4s(s-a)}{bc} \cdot \frac{bc}{(b+c)^2} = 4\cos^2\frac{A}{2} \cdot \frac{bc}{(b+c)^2}$$

$$\therefore \cos\theta = \frac{2\sqrt{bc}}{b+c} \cos\frac{A}{2}$$

11. If $a = (b+c)\cos\theta$, show that $\sin\theta = \frac{2\sqrt{bc}}{b+c} \cos\frac{A}{2}$.

Sol: Given $a = (b+c)\cos\theta \Rightarrow \cos\theta = \frac{a}{b+c}$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{a}{b+c}\right)^2 = 1 - \frac{a^2}{(b+c)^2} = \frac{(b+c)^2 - a^2}{(b+c)^2}$$

$$= \frac{(b+c+a) - (b+c-a)}{(b+c)^2} = \frac{(2s) - (2s-a-a)}{(b+c)^2} = \frac{(2s) - 2(s-a)}{(b+c)^2}$$

$$= 4 \cdot \frac{s(s-a)}{bc} \cdot \frac{bc}{(b+c)^2} = 4 \cdot \cos^2\frac{A}{2} \cdot \frac{bc}{(b+c)^2}$$

$$\sin^2\theta = 4 \frac{bc}{(b+c)^2} \cdot \cos^2\frac{A}{2}$$

$$\therefore \sin\theta = \frac{2\sqrt{bc}}{b+c} \cos\frac{A}{2}$$

12. If $a = (b-c)\sec\theta$, show that $\tan\theta = \frac{2\sqrt{bc}}{b-c} \sin\frac{A}{2}$.

Sol: $a = (b-c)\sec\theta \Rightarrow \sec\theta = \frac{a}{b-c}$

$$\tan^2\theta = \sec^2\theta - 1 = \left(\frac{a}{b-c}\right)^2 - 1$$

$$\begin{aligned}\tan^2\theta &= \frac{a^2 - (b - c)^2}{(b - c)^2} = \frac{(a + b - c)(a - b + c)}{(b - c)^2} \\ &= \frac{2(s - c) \cdot 2(s - b)}{(b - c)^2} = \frac{4(s - c)(s - b)}{bc} \cdot \frac{bc}{(b - c)^2}\end{aligned}$$

$$\tan^2\theta = 4 \frac{bc}{(b - c)^2} \sin^2 \frac{A}{2}$$

$$\therefore \tan\theta = \frac{2\sqrt{bc}}{b - c} \sin \frac{A}{2} \quad \therefore \tan\theta = \frac{2\sqrt{bc}}{b - c} \sin \frac{A}{2}$$

13. Prove that $\cot A + \cot B + \cot C = \frac{a^2 + b^2 + c^2}{4\Delta}$.

Sol: $\cot A + \cot B + \cot C = \sum \cot A = \sum \frac{\cos A}{\sin A}$

$$\begin{aligned}&= \sum \left(\frac{\left(\frac{b^2 + c^2 - a^2}{2bc} \right)}{\sin A} \right) = \sum \left(\frac{b^2 + c^2 - a^2}{2bc \sin A} \right) \\ &= \sum \frac{b^2 + c^2 - a^2}{4\Delta} \quad [\because \Delta = \frac{1}{2} bc \sin A] \\ &= \frac{b^2 + c^2 - a^2}{4\Delta} + \frac{c^2 + a^2 - b^2}{4\Delta} + \frac{a^2 + b^2 - c^2}{4\Delta} \\ &= \frac{b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2}{4\Delta} = \frac{a^2 + b^2 + c^2}{4\Delta} = \text{RHS} \\ \therefore \cot A + \cot B + \cot C &= \frac{a^2 + b^2 + c^2}{4\Delta}\end{aligned}$$

14. In $\triangle ABC$, $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$ then show that $\angle C = 60^\circ$.

Sol: $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

$$\frac{b+c+a+c}{(a+c)(b+c)} = \frac{3}{a+b+c}$$

$$(a+b+2c)(a+b+c) = 3(a+c)(b+c)$$

$$a^2 + ab + ac + ba + b^2 + bc + 2ac + 2bc + 2c^2 = 3[ab + ac + bc + c^2]$$

$$a^2 + b^2 - c^2 = ab$$

$$2ab\cos C = ab \quad (\because \text{from Cosine rule})$$

$$2\cos C = 1$$

$$\cos C = \frac{1}{2} = \cos 60^\circ$$

$$\therefore \angle C = 60^\circ$$

- 15.** In $\triangle ABC$, if $a\cos A = b\cos B$ then show that triangle is isosceles (or) right angle triangle.

Sol: $a\cos A = b\cos B$

$$2R\sin A\cos A = 2R\sin B\cos B \quad (\because \text{From Sine rule})$$

$$\sin 2A = \sin 2B = \sin(180 - 2B)$$

$$2A = 2B \quad (\text{లేదా}) \quad 2A = 180 - 2B$$

$$A = B \quad (\text{లేదా}) \quad A = 90 - B$$

$$A = B \quad (\text{లేదా}) \quad A + B = 90^\circ$$

$$\Rightarrow a = b \quad (\text{లేదా}) \quad \angle C = 90^\circ$$

$$\therefore \triangle ABC \text{ is isosceles or right angle triangle.}$$

- 16.** If $a : b : c = 7 : 8 : 9$ then find $\cos A : \cos B : \cos C$.

Sol: $a : b : c = 7 : 8 : 9$

$$\frac{a}{7} = \frac{b}{8} = \frac{c}{9} = k$$

$$a = 7k; b = 8k; c = 9k$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{64k^2 + 81k^2 - 49k^2}{2(8k)(9k)} = \frac{96k^2}{144k^2} = \frac{2}{3}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{49k^2 + 81k^2 - 64k^2}{2(7k)(9k)} = \frac{66k^2}{126k^2} = \frac{11}{21}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49k^2 + 64k^2 - 81k^2}{2(7k)(8k)} = \frac{32k^2}{112k^2} = \frac{2}{7}$$

$$\therefore \cos A : \cos B : \cos C = \frac{2}{3} : \frac{11}{21} : \frac{2}{7} = \left(\frac{2 \times 7}{3 \times 7} \right) : \frac{11}{21} : \left(\frac{2 \times 3}{7 \times 3} \right)$$

$$\cos A : \cos B : \cos C = 14 : 11 : 6$$

- 17.** In $\triangle ABC$, P_1, P_2, P_3 are altitudes, then show that $\frac{1}{P_1^2} + \frac{1}{P_2^2} + \frac{1}{P_3^2} = \frac{\cot A + \cot B + \cot C}{\Delta}$.

Sol: In $\triangle ABC$, AD, BE, CF are altitudes.

$$AD = P_1, BE = P_2, CF = P_3$$

$$\Delta = \frac{1}{2} BC \times AD = \frac{1}{2} CA \times BE = \frac{1}{2} AB \times CF$$

$$\Delta = \frac{1}{2} a.P_1 = \frac{1}{2} b.P_2 = \frac{1}{2} c.P_3$$

$$\therefore P_1 = \frac{2\Delta}{a}; P_2 = \frac{2\Delta}{b}; P_3 = \frac{2\Delta}{c}$$

$$\frac{1}{P_1^2} + \frac{1}{P_2^2} + \frac{1}{P_3^2} = \frac{a^2}{4\Delta^2} + \frac{b^2}{4\Delta^2} + \frac{c^2}{4\Delta^2} = \frac{a^2 + b^2 + c^2}{4\Delta^2}$$

$$= \frac{1}{\Delta} \left(\frac{a^2 + b^2 + c^2}{4\Delta} \right) = \frac{1}{\Delta} (\cot A + \cot B + \cot C) \quad (\because \text{From problem 13})$$

$$\therefore \frac{1}{P_1^2} + \frac{1}{P_2^2} + \frac{1}{P_3^2} = \frac{\cot A + \cot B + \cot C}{\Delta}$$

18. Show that $\sum a \cot A = 2(R + r)$.

Sol:
$$\begin{aligned} \text{LHS} &= \sum a \cot A = \sum 2R \sin A \frac{\cos A}{\sin A} = \sum 2R \cos A \\ &= 2R (\cos A + \cos B + \cos C) \\ &= 2R \left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) \end{aligned}$$

$$(\because \text{from transformations } \cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2})$$

$$\begin{aligned} &= 2 \left[R + 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right] \\ &= 2[R + r] \end{aligned}$$

$$\therefore \sum a \cot A = 2(R + r)$$

19. Prove that $r(r_1 + r_2 + r_3) = ab + bc + ca - s^2$.

Sol:
$$\begin{aligned} \text{LHS} &= r(r_1 + r_2 + r_3) = \frac{\Delta}{s} \left(\frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} \right) \\ &= \frac{\Delta^2}{s} \left(\frac{(s-b)(s-c) + (s-a)(s-c) + (s-a)(s-b)}{(s-a)(s-b)(s-c)} \right) \\ &= \frac{\Delta^2 [s^2 - s(b+c) + s^2 - s(a+c) + s^2 - s(a+b) + bc + ac + ab]}{\Delta^2} \end{aligned}$$

$$= 3s^2 - 2s(a + b + c) + ab + bc + ca$$

$$= 3s^2 - 2s(2s) + ab + bc + ca$$

$$= ab + bc + ca - s^2 = \text{RHS}$$

$$\therefore r(r_1 + r_2 + r_3) = ab + bc + ca - s^2$$

20. In $\triangle ABC$ show that $r_1 + r_2 + r_3 - r = 4R$.

Sol: $r_1 + r_2 + r_3 - r$

$$= 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + 4R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} + 4R \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$$

$$- 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= 4R \cos \frac{C}{2} \left[\sin \frac{A}{2} \cos \frac{B}{2} + \cos \frac{A}{2} \sin \frac{B}{2} \right] + 4R \sin \frac{C}{2} \left[\cos \frac{A}{2} \cos \frac{B}{2} - \sin \frac{A}{2} \sin \frac{B}{2} \right]$$

$$= 4R \cos \frac{C}{2} \sin \left(\frac{A}{2} + \frac{B}{2} \right) + 4R \sin \frac{C}{2} \cos \left(\frac{A}{2} + \frac{B}{2} \right)$$

$$= 4R \cos \frac{C}{2} \sin \left(\frac{A+B}{2} \right) + 4R \sin \frac{C}{2} \cos \left(\frac{A+B}{2} \right)$$

$$= 4R \sin \left(\frac{A+B}{2} + \frac{C}{2} \right) = 4R \sin \left(\frac{A+B+C}{2} \right) = 4R \sin \left(\frac{\pi}{2} \right)$$

$$= 4R(1) = 4R = \text{RHS}$$

$$\therefore r_1 + r_2 + r_3 - r = 4R$$

21. In $\triangle ABC$ prove that $r + r_1 + r_2 - r_3 = 4R \cos C$.

Sol: LHS = $r + r_1 + r_2 - r_3$

$$4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + 4R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$$

$$- 4R \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$$

$$= 4R \sin \frac{A}{2} \left[\sin \frac{B}{2} \sin \frac{C}{2} + \cos \frac{B}{2} \cos \frac{C}{2} \right] + 4R \cos \frac{A}{2} \left[\sin \frac{B}{2} \cos \frac{C}{2} - \cos \frac{B}{2} \sin \frac{C}{2} \right]$$

$$= 4R \sin \frac{A}{2} \cos \left(\frac{B}{2} - \frac{C}{2} \right) + 4R \cos \frac{A}{2} \sin \left(\frac{B}{2} - \frac{C}{2} \right)$$

$$= 4R \left[\sin \frac{A}{2} \cos \left(\frac{B-C}{2} \right) + \cos \frac{A}{2} \sin \left(\frac{B-C}{2} \right) \right]$$

$$= 4R \cdot \sin\left(\frac{A}{2} + \frac{B-C}{2}\right) = 4R \sin\left(\frac{A+B-C}{2}\right)$$

$$= 4R \sin\left(\frac{\pi - C - C}{2}\right) = 4R \sin\left(\frac{\pi}{2} - C\right)$$

$$= 4R \cos C = \text{RHS}$$

$$\therefore r + r_1 + r_2 - r_3 = 4R \cos C$$

22. Prove that $\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{abc}{\Delta^3} = \frac{4R}{r^2 s^2}$.

Sol: $\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right)$

$$= \left(\frac{s}{\Delta} - \frac{s-a}{\Delta}\right)\left(\frac{s}{\Delta} - \frac{s-b}{\Delta}\right)\left(\frac{s}{\Delta} - \frac{s-c}{\Delta}\right) = \left(\frac{s-s+a}{\Delta}\right)\left(\frac{s-s+b}{\Delta}\right)\left(\frac{s-s+c}{\Delta}\right)$$

$$= \left(\frac{a}{\Delta}\right)\left(\frac{b}{\Delta}\right)\left(\frac{c}{\Delta}\right) = \frac{abc}{\Delta^3} = \frac{4R\Delta}{\Delta^3} = \frac{4R}{\Delta^2} \quad \left[\because \Delta = \frac{abc}{4R}, abc = 4R\Delta\right]$$

$$= \frac{4R}{(rs)^2} = \frac{4R}{r^2 s^2}$$

$$\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{abc}{\Delta^3} = \frac{4R}{r^2 s^2}$$

23. Show that $\sum \frac{r_i}{(s-b)(s-c)} = \frac{3}{r}$.

Sol: $\text{LHS} = \sum \frac{r_i}{(s-b)(s-c)} = \sum \frac{\Delta}{(s-a)(s-b)(s-c)} \quad \left[\because r_i = \frac{\Delta}{s-a}\right]$

$$= \sum \frac{\Delta}{\left(\frac{\Delta^2}{s}\right)} \quad \left[\because \Delta^2 = s(s-a)(s-b)(s-c)\right]$$

$$= \sum \frac{s\Delta}{\Delta^2} = \sum \frac{s}{\Delta} = \frac{s}{\Delta} + \frac{s}{\Delta} + \frac{s}{\Delta} = \frac{3s}{\Delta}$$

$$= 3\left(\frac{s}{\Delta}\right) = 3\left(\frac{1}{r}\right) = \frac{3}{r} = \text{RHS}$$

$$\therefore \sum \frac{r_i}{(s-b)(s-c)} = \frac{3}{r}$$

23. Show that $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$.

Sol:
$$\begin{aligned} \text{LHS} &= \cos A + \cos B + \cos C = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \cos C \\ &= 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2} \quad \left(\because \frac{A+B}{2} = 90 - \frac{C}{2}, \cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}\right) \\ &= 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}\right] \\ &= 1 + 2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right] \\ &= 1 + 2\sin\frac{C}{2}\left[2\sin\frac{A}{2}\sin\frac{B}{2}\right] \\ &= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \\ &= 1 + \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R} = 1 + \frac{r}{R} = \text{RHS} \end{aligned}$$

$\therefore \cos A + \cos B + \cos C = 1 + \frac{r}{R}$

25. Show that $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2 + \frac{r}{2R}$.

Sol:
$$\begin{aligned} \cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} &= \cos^2\frac{A}{2} + 1 - \sin^2\frac{B}{2} + \cos^2\frac{C}{2} \\ &= 1 + \left(\cos^2\frac{A}{2} - \sin^2\frac{B}{2}\right) + \cos^2\frac{C}{2} = 1 + \cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \cos^2\frac{C}{2} \\ &= 1 + \sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + 1 - \sin^2\frac{C}{2} \quad \left[\begin{array}{l} \because \frac{A+B}{2} = 90 - \frac{C}{2} \\ \cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2} \end{array}\right] \\ &= 2 + \sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin^2\frac{C}{2}\right] \\ &= 2 + \sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right] \end{aligned}$$

$$\begin{aligned}
&= 2 + \sin \frac{C}{2} \left[2 \sin \frac{A}{2} \sin \frac{B}{2} \right] \\
&= 2 + \sin \frac{C}{2} \left[2 \sin \frac{A}{2} \sin \frac{B}{2} \right] = 2 + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\
&= 2 + \frac{(2R) \cdot 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{2R} = 2 + \frac{4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{2R} \\
&= 2 + \frac{r}{2R} = \text{RHS} \\
\therefore \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} &= 2 + \frac{r}{2R}
\end{aligned}$$

26. In $\triangle ABC$, P_1, P_2, P_3 are the altitudes drawn from the vertices A, B, C to the opposite sides, then show that

$$(i) \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = \frac{1}{r} \quad (ii) \frac{1}{P_1} + \frac{1}{P_2} - \frac{1}{P_3} = \frac{1}{r_3} \quad (iii) P_1 P_2 P_3 = \frac{(abc)^2}{8R^3} = \frac{8\Delta^3}{abc}$$

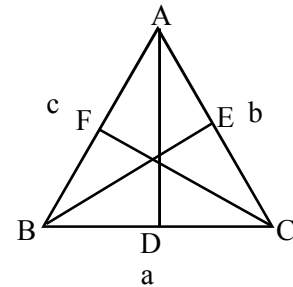
Sol: In $\triangle ABC$

$AD = P_1, BE = P_2, CF = P_3$ are altitudes.

$$\Delta = \frac{1}{2} a \cdot P_1 = \frac{1}{2} b \cdot P_2 = \frac{1}{2} c \cdot P_3$$

$$2\Delta = aP_1, 2\Delta = bP_2, 2\Delta = cP_3$$

$$P_1 = \frac{2\Delta}{a}; P_2 = \frac{2\Delta}{b}; P_3 = \frac{2\Delta}{c}$$



$$(i) \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} + \frac{c}{2\Delta} = \frac{a+b+c}{2\Delta} = \frac{2s}{2\Delta} = \frac{s}{\Delta} = \frac{1}{r}$$

$$(ii) \frac{1}{P_1} + \frac{1}{P_2} - \frac{1}{P_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} - \frac{c}{2\Delta} = \frac{a+b-c}{2\Delta} = \frac{2s-c-c}{2\Delta} = \frac{2(s-c)}{2\Delta} = \frac{s-c}{\Delta} = \frac{1}{r_3}$$

$$(iii) P_1 P_2 P_3 = \frac{2\Delta}{a} \times \frac{2\Delta}{b} \times \frac{2\Delta}{c} = \frac{8\Delta^3}{abc}$$

$$= \frac{8\left(\frac{abc}{4R}\right)^3}{abc} = \frac{8(abc)^3}{(64R^3)abc} = \frac{(abc)^2}{8R^3} \therefore P_1 P_2 P_3 = \frac{(abc)^2}{8R^3} = \frac{8\Delta^3}{abc}$$

27. If $a = 13$, $b = 14$, $c = 15$, then show that $R = \frac{65}{8}$, $r = 4$, $r_1 = \frac{21}{2}$, $r_2 = 12$ and $r_3 = 14$.

Sol: $a = 13$, $b = 14$, $c = 15$

$$2s = a + b + c = 13 + 14 + 15 = 42$$

$$s = 21$$

$$\Delta^2 = s(s-a)(s-b)(s-c) = 21(21-13)(21-14)(21-15) \\ = (21)(8)(7)(6)$$

$$\Delta = \sqrt{21 \times 8 \times 7 \times 6} = \sqrt{7 \times 3 \times 2 \times 2 \times 2 \times 7 \times 2 \times 3} = 7 \times 3 \times 2 \times 2 = 84$$

$$\Delta = 84$$

$$R = \frac{abc}{4\Delta} = \frac{13 \cdot 14 \cdot 15}{4 \cdot 84} = \frac{65}{8}$$

$$r = \frac{\Delta}{s} = \frac{84}{21} = 4$$

$$r_1 = \frac{\Delta}{s-a} = \frac{84}{21-13} = \frac{84}{8} = \frac{21}{2}$$

$$r_2 = \frac{\Delta}{s-b} = \frac{84}{21-14} = \frac{84}{7} = 12$$

$$r_3 = \frac{\Delta}{s-c} = \frac{84}{21-15} = \frac{84}{6} = 14$$

$$\therefore R = \frac{65}{8}, r = 4, r_1 = \frac{21}{2}, r_2 = 12, r_3 = 14$$

28. $r_1 = 2$, $r_2 = 3$, $r_3 = 6$ and $r = 1$, then show that $a = 3$, $b = 4$, $c = 5$.

Sol: $r_1 = 2$, $r_2 = 3$, $r_3 = 6$, $r = 1$

$$\Delta^2 = r r_1 r_2 r_3 = (1)(2)(3)(6) = 36$$

$$\Delta = 6$$

$$r = \frac{\Delta}{s} \Rightarrow s = \frac{\Delta}{r} = \frac{6}{1} = 6$$

$$s = 6$$

$$r_1 = \frac{\Delta}{s-a} \Rightarrow s-a = \frac{\Delta}{r_1} = \frac{6}{2} = 3$$

$$s-a = 3$$

$$6-a = 3$$

$$a = 3$$

$$r_2 = \frac{\Delta}{s-b} \Rightarrow s-b = \frac{\Delta}{r_2} = \frac{6}{3} = 2$$

$$s-b = 2 \Rightarrow 6-b = 2$$

$$b = 4$$

$$r_3 = \frac{\Delta}{s - c} \Rightarrow s - c = \frac{\Delta}{r_3} = \frac{6}{6} = 1$$

$$6 - c = 1$$

$$c = 5$$

$$\therefore a = 3, b = 4, c = 5$$

29. In $\triangle ABC$ $r_1 = 8, r_2 = 12, r_3 = 24$ then find the values of a, b, c .

Sol: $r_1 = 8, r_2 = 12, r_3 = 24$

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{3+2+1}{24} = \frac{6}{24} = \frac{1}{4}$$

$$r = 4$$

$$\Delta^2 = r r_1 r_2 r_3 = (4)(8)(12)(24) = 4 \times 8 \times 12 \times 12 \times 2 = 12 \times 8 \times 12 \times 8$$

$$\Delta = 12 \times 8 = 96$$

$$\Delta = 96$$

$$\Delta = rs \Rightarrow s = \frac{\Delta}{r} = \frac{96}{4} = 24$$

$$s = 24$$

$$r_1 = \frac{\Delta}{s - a} \Rightarrow s - a = \frac{\Delta}{r_1} = \frac{96}{8} = 12$$

$$s - a = 12$$

$$24 - a = 12 \quad a = 12$$

$$r_2 = \frac{\Delta}{s - b} \Rightarrow s - b = \frac{\Delta}{r_2} = \frac{96}{12} = 8$$

$$s - b = 8$$

$$24 - b = 8$$

$$b = 16$$

$$r_3 = \frac{\Delta}{s - c} \Rightarrow s - c = \frac{\Delta}{r_3} = \frac{96}{24} = 4$$

$$s - c = 4$$

$$24 - c = 4$$

$$c = 20$$

$$\therefore a = 12, b = 16, c = 20$$

30. Show that $\frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2 + b^2 + c^2}{\Delta^2}$.

Sol:
$$\begin{aligned} \frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} &= \frac{s^2}{\Delta^2} + \frac{(s-a)^2}{\Delta^2} + \frac{(s-b)^2}{\Delta^2} + \frac{(s-c)^2}{\Delta^2} \\ &= \frac{1}{\Delta^2} [s^2 + (s-a)^2 + (s-b)^2 + (s-c)^2] \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\Delta^2} [s^2 + s^2 + a^2 - 2as + s^2 + b^2 - 2bs + s^2 + c^2 - 2cs] \\
 &= \frac{1}{\Delta^2} [4s^2 - 2s(a + b + c) + a^2 + b^2 + c^2] \\
 &= \frac{1}{\Delta^2} [4s^2 - 2s(2s) + a^2 + b^2 + c^2] \\
 &= \frac{a^2 + b^2 + c^2}{\Delta^2} = \text{RHS}
 \end{aligned}$$

$$\therefore \frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2 + b^2 + c^2}{\Delta^2}$$

31. Show that $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{1}{r} - \frac{1}{2R}$.

$$\begin{aligned}
 \text{Sol: LHS} &= \frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{1}{abc} [ar_1 + br_2 + cr_3] \\
 &= \frac{1}{abc} \sum ar_1 = \frac{1}{abc} \sum 2R \sin A \cdot \tan \frac{A}{2} \\
 &= \frac{1}{abc} \sum 2R \cdot 2 \sin \frac{A}{2} \cos \frac{A}{2} \cdot \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \frac{1}{abc} \cdot s \sum 4R \sin^2 \frac{A}{2} \\
 &= \frac{4RS}{abc} \sum \sin^2 \frac{A}{2} = \frac{s}{\Delta} \sum \frac{1 - \cos A}{2} \left[\because \Delta = \frac{abc}{4R} \right] \\
 &= \frac{1}{r} \left[\frac{1 - \cos A}{2} + \frac{1 - \cos B}{2} + \frac{1 - \cos C}{2} \right] \\
 &= \frac{1}{r} \left[\frac{1 - \cos A + 1 - \cos B + 1 - \cos C}{2} \right] = \frac{1}{r} \left[\frac{3 - (\cos A + \cos B + \cos C)}{2} \right] \\
 &= \frac{1}{2r} [3 - (\cos A + \cos B + \cos C)] \quad \left[\because \cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right] \\
 &= \frac{1}{2r} \left[3 - \left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) \right] \\
 &= \frac{1}{2r} \left[2 - 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right] \\
 &= \frac{2}{2r} - \frac{4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{2r} = \frac{1}{r} - \frac{4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{2rR} = \frac{1}{r} - \frac{r}{2rR} \\
 &= \frac{1}{r} - \frac{1}{2R} \text{ RHS}
 \end{aligned}$$

